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Higher Order Langevin Monte Carlo Algorithm

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Abstract

A new (unadjusted) Langevin Monte Carlo (LMC) algorithm with improved rates in total variation and in Wasserstein distance is presented. All these are obtained in the context of sampling from a target distribution π that has a density on \mathbb{R}^d known up to a normalizing constant. Crucially, the Langevin SDE associated with the target distribution π is assumed to have a locally Lipschitz drift coefficient such that its second derivative is locally Hölder continuous with exponent $\beta \in (0, 1]$. Non-asymptotic bounds are obtained for the convergence to stationarity of the new sampling method with convergence rate $1 + \beta/2$ in Wasserstein distance, while it is shown that the rate is 1 in total variation even in the absence of convexity. Finally, in the case of Lipschitz gradient, explicit constants are provided.

1 Introduction

In Bayesian statistics and machine learning, one challenge, which has attracted substantial attention in recent years due to its high importance in data-driven applications, is the creation of algorithms which can efficiently sample from a high-dimensional target probability distribution π . In particular, its smooth version assumes that there exists a density on \mathbb{R}^d , also denoted by π , such that

$$\pi(x) = e^{-U(x)} / \int_{\mathbb{R}^d} e^{-U(y)} dy,$$

with $\int_{\mathbb{R}^d} e^{-U(y)} dy < \infty$, where U is typically continuously differentiable. Within such a setting, consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, then the Langevin SDE associated with π is defined by

$$dx_t = -\nabla U(x_t)dt + \sqrt{2}dw_t, \quad (1)$$

where $(w_t)_{t \geq 0}$ is a d -dimensional Brownian motion. It is a classical result that under mild conditions, the SDE (1) admits π as its unique invariant measure. The corresponding numerical scheme of the Langevin SDE obtained by using the Euler-Maruyama (Milstein) method yields the unadjusted Langevin algorithm (ULA), known also as the Langevin Monte Carlo (LMC), which has been well studied in the literature. For a globally Lipschitz ∇U , the non-asymptotic bounds in total variation and Wasserstein distance between the n -th iteration of the ULA algorithm and π have been provided in [11], [12] and [10]. As for the case of superlinear ∇U , the difficulty arises from the fact that ULA is unstable (see [23]), and its Metropolis adjusted version, MALA, loses some of its appealing properties as discussed in [7] and demonstrated numerically in [2]. However, recent research has developed new types of explicit numerical schemes for SDEs with superlinear coefficients, and it has been shown

in [15], [17], [16], [18], [13], [19], that the tamed Euler (Milstein) scheme converges to the true solution of the SDE (1) in \mathcal{L}^p on any given finite time horizon with optimal rate. This progress led to the creation of the tamed unadjusted Langevin algorithm (TULA) in [2], where the aforementioned convergence results are extended to an infinite time horizon and, moreover, one obtains rate of convergence results in total variation and in Wasserstein distance.

In this article, a new higher order LMC algorithm is constructed, which is based on the order 1.5 scheme (3) of the SDE (1). One observes that the scheme (2) coincides with (3) in distribution, and the latter is obtained using an Itô-Taylor expansion (see [22]). By extending the techniques used in [2] and [20], it can be shown that the scheme (2) has a unique invariant measure π_γ , where $\gamma \in (0, 1)$ is the stepsize of the scheme, and one obtains the improved convergence results between π_γ and the target distribution π . More precisely, assume the potential U is three times differentiable, and its third derivative is locally Hölder continuous with exponent $\beta \in (0, 1]$. Then, under certain conditions (specified in Section 2), Theorem 1 and 2 state that the rate of convergence between the n -th iteration of the new algorithm and the target measure π is $1 + \beta/2$ in Wasserstein distance, whereas the rate is 1 in total variation. Here, one notes that these results are obtained in the context of having superlinear ∇U . To the best of the authors' knowledge, these are the first such results which provide a higher rate of convergence in Wasserstein distance compared to the existing literature. As for the total variation distance, [11] proves that the rate of convergence is 1 for the case of a strongly convex U , whereas our result yields the same convergence rate without assuming convexity.

An important feature of the newly proposed, higher order LMC is its computational efficiency. It can be seen in key paradigms of the area, such as sampling from a high-dimensional Gaussian distribution, since one obtains with limited additional computational effort, that the distribution of the algorithm converges faster to the target distribution π compared to existing algorithms (see [9], [11] and references therein). In other words, it takes fewer steps for the error to be less than a given ε , where the error is measured using the notion of a 2-Wasserstein distance between the n -th iteration of the algorithm (3) and π . To illustrate this point, consider the multivariate Gaussian as the target distribution π defined by

$$x \mapsto \pi(x) \propto e^{-\frac{1}{2}x^\top \Sigma^{-1}x},$$

where $\Sigma \in \mathbb{R}^{d \times d}$ is the covariance matrix. Then, for any $x \in \mathbb{R}^d$

$$U(x) = -\frac{1}{2}x^\top \Sigma^{-1}x, \quad \nabla U(x) = \Sigma^{-1}x, \quad \nabla^2 U(x) = \Sigma^{-1}.$$

As the third derivative of U is zero in this case, the proposed algorithm (3) can be obtained by just adding two more terms on the ULA algorithm (the corresponding scheme (2) can be obtained accordingly), which are easily computed. If the precision level at n -th iteration is set to be ε , i.e. $W_2(\delta_x R_\gamma^n, \pi) \leq \varepsilon$, then according to Corollary 1, by letting

$$e^{-mn\gamma} \left(2|x - x^*|^2 + \frac{2d}{m} \right) \leq \frac{\varepsilon^2}{2}, \quad \tilde{C}\gamma^3 \leq \frac{\varepsilon^2}{2},$$

one obtains $n \geq \left((2\tilde{C})^{\frac{1}{3}}/m\varepsilon^{\frac{2}{3}} \right) \log(4(|x - x^*|^2 + d/m)/\varepsilon^2)$, where $\tilde{C} > 0$ is some constant that is proportional to d and x^* is the unique minimizer of U . Thus, the algorithm (3) requires much fewer steps to reach a certain precision level compared to results in [9] and [11]. However, one notes that for the case of a Lipschitz gradient with nonzero L_2 and L in assumptions **H5** and **H6** (see below), the dependency of the Wasserstein bound on the dimension increases from d to d^2 .

We conclude this section by introducing some notation. The Euclidean norm of a vector $b \in \mathbb{R}^d$ and the Hilbert-Schmidt norm of a matrix $\sigma \in \mathbb{R}^{d \times m}$ are denoted by $|b|$ and $|\sigma|$ respectively. σ^\top is the transpose matrix of σ and \mathbf{I}_d is the $d \times d$ identity matrix. The i -th element of b and (i, j) -th element of σ are denoted respectively by $b^{(i)}$ and $\sigma^{(i, j)}$, for every $i = 1, \dots, d$ and $j = 1, \dots, d$. In addition, denote by $[a]$ the integer part of a positive real number a , and $\lceil a \rceil = [a] + 1$. The inner product of two vectors $x, y \in \mathbb{R}^d$ is denoted by xy . For all $x \in \mathbb{R}^d$ and $M > 0$, denote by $B(x, M)$ (respectively $\bar{B}(x, M)$) the open (respectively close) ball centered at x with radius M . Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a twice continuously differentiable function. Denote by ∇f , $\nabla^2 f$ and Δf the gradient of f , the Hessian of f and the Laplacian of f respectively. Consider a twice continuously differentiable function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$. The term $\nabla^2 g$ is an array of d Hessian matrices, one for each component of g , i.e. $\nabla^2 g = [\nabla^2 g^{(1)}, \dots, \nabla^2 g^{(d)}]$. For all $x \in \mathbb{R}^d$, the norm $|\nabla^2 g(x)|$ is defined by $\sqrt{\sum_{i=1}^d |\nabla^2 g^{(i)}(x)|^2}$. Denote by $\vec{\Delta} g$ the vector Laplacian of g , i.e., for all $x \in \mathbb{R}^d$, $\vec{\Delta} g(x)$ is a vector in \mathbb{R}^d whose i -th entry is $\sum_{u=1}^d \frac{\partial^2 g^{(i)}}{\partial x^{(u)} \partial x^{(u)}}(x)$. For $m, m' \in \mathbb{N}^*$, define

$$C_{\text{poly}}(\mathbb{R}^m, \mathbb{R}^{m'}) = \left\{ P \in C(\mathbb{R}^m, \mathbb{R}^{m'}) \mid \exists C_q, q \geq 0, \forall x \in \mathbb{R}^d, |P(x)| \leq C_q(1 + |x|^q) \right\}.$$

For any $t \geq 0$, denote by $\mathcal{C}([0, t], \mathbb{R}^d)$ the space of continuous \mathbb{R}^d -valued paths defined on the time interval $[0, t]$.

Let μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and f be a μ -integrable function, define $\mu(f) = \int_{\mathbb{R}^d} f(x) d\mu(x)$. Given a Markov kernel R on \mathbb{R}^d and a function f integrable under $R(x, \cdot)$, denote by $Rf(x) = \int_{\mathbb{R}^d} f(y) R(x, dy)$. Let $V : \mathbb{R}^d \rightarrow [1, \infty)$ be a measurable function. The V -total variation distance between μ and ν is defined as $\|\mu - \nu\|_V = \sup_{|f| \leq V} |\mu(f) - \nu(f)|$. If $V = 1$, then $\|\cdot\|_V$ is the total variation denoted by $\|\cdot\|_{TV}$. Let μ and ν be two probability measures on a state space Ω with a given σ -algebra. If $\mu \ll \nu$, we denote by $d\mu/d\nu$ the Radon-Nikodym derivative of μ w.r.t. ν . Then, the Kullback-Leibler divergence of μ w.r.t. ν is given by

$$\text{KL}(\mu|\nu) = \int_{\Omega} \frac{d\mu}{d\nu} \log \left(\frac{d\mu}{d\nu} \right) d\nu.$$

We say that ζ is a transference plan of μ and ν if it is a probability measure on $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d))$ such that for any Borel set A of \mathbb{R}^d , $\zeta(A \times \mathbb{R}^d) = \mu(A)$ and $\zeta(\mathbb{R}^d \times A) = \nu(A)$. We denote by $\Pi(\mu, \nu)$ the set of transference plans of μ and ν . Furthermore, we say that a couple of \mathbb{R}^d -random variables (X, Y) is a coupling of μ and ν if there exists $\zeta \in \Pi(\mu, \nu)$ such that (X, Y) are distributed according to ζ . For two probability measures μ and ν , the Wasserstein distance of order $p \geq 1$ is defined as

$$W_p(\mu, \nu) = \left(\inf_{\zeta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\zeta(\mu, \nu) \right)^{1/p}.$$

By Theorem 4.1 in [4], for all probability measures μ, ν on \mathbb{R}^d , there exists a transference plan $\zeta^* \in \Pi(\mu, \nu)$ such that for any coupling (X, Y) distributed according to ζ^* , we have $W_p(\mu, \nu) = \mathbb{E}[|X - Y|^p]^{1/p}$.

2 Main results

Assume U is three times continuously differentiable. The following conditions are stated:

H1 $\liminf_{|x| \rightarrow +\infty} |\nabla U(x)| = +\infty$, and $\liminf_{|x| \rightarrow +\infty} \frac{x \nabla U(x)}{|x| |\nabla U(x)|} > 0$.

H2 There exists $L > 0$, $\rho \geq 2$, and $\beta \in (0, 1]$, such that for all $x, y \in \mathbb{R}^d$,

$$|\nabla^2(\nabla U)(x) - \nabla^2(\nabla U)(y)| \leq L(1 + |x| + |y|)^{\rho-2}|x - y|^\beta.$$

H3 U is strongly convex, i.e. there exists $m > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$(x - y) (\nabla U(x) - \nabla U(y)) \geq m|x - y|^2.$$

Remark 1. Unless otherwise specified, the constants $C, K > 0$ may take different values at different places, but these are always independent of the step size $\gamma \in (0, 1)$.

Remark 2. Assume **H2** holds, then there exists a constant $K > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\nabla^2(\nabla U)(x)| \leq K(1 + |x|)^{\rho-2+\beta},$$

moreover, there exists a constant $K > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\nabla^2 U(x) - \nabla^2 U(y)| \leq K(1 + |x| + |y|)^{\rho-2+\beta}|x - y|,$$

which implies,

$$|\nabla^2 U(x)| \leq K(1 + |x|)^{\rho-1+\beta}.$$

Furthermore, there exists a constant $K > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\nabla U(x) - \nabla U(y)| \leq K(1 + |x| + |y|)^{\rho-1+\beta}|x - y|,$$

$$|\nabla U(x)| \leq K(1 + |x|)^{\rho+\beta}.$$

For any $x \in \mathbb{R}^d$, denote by

$$\nabla U_\gamma(x) = \frac{\nabla U(x)}{(1 + \gamma^{3/2}|\nabla U(x)|^{3/2})^{2/3}}, \quad \nabla^2 U_\gamma(x) = \frac{\nabla^2 U(x)}{1 + \gamma|\nabla^2 U(x)|},$$

$$(\nabla U \nabla^2 U)_\gamma(x) = \frac{\nabla^2 U(x) \nabla U(x)}{1 + \gamma|x||\nabla^2 U(x)||\nabla U(x)|}, \quad \vec{\Delta}(\nabla U)_\gamma(x) = \frac{\vec{\Delta}(\nabla U)(x)}{1 + \gamma^{1/2}|x||\vec{\Delta}(\nabla U)(x)|}.$$

The taming is chosen in a way such that (7) holds, which leads to the finiteness of exponential moments of the scheme due to the Log-Sobolev inequality, while the desired rate of convergence of the tamed drift to the original drift can be obtained.

Remark 3. There exists a constant $K > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\nabla U_\gamma(x)| \leq K\gamma^{-1}, \quad |\nabla^2 U_\gamma(x)| \leq K\gamma^{-1},$$

$$|(\nabla U \nabla^2 U)_\gamma(x)| \leq K\gamma^{-1}, \quad |\vec{\Delta}(\nabla U)_\gamma(x)| \leq K\gamma^{-1/2}.$$

Note that in Remark 3, the upper bound for $|(\nabla U \nabla^2 U)_\gamma(x)|$ is $K\gamma^{-1}$ since for $|x| \leq 1$, one estimates the term using **H2**, while for $|x| > 1$, the bound is obtained by a straightforward calculation using the taming defined above.

The new higher order LMC algorithm, which is the tamed order 1.5 scheme of the SDE (1), has also the following representation, for any $n \in \mathbb{N}$,

$$\bar{X}_{n+1} = \bar{X}_n + \mu_\gamma(\bar{X}_n)\gamma + \sigma_\gamma(\bar{X}_n)\sqrt{\gamma}Z_{n+1}, \quad (2)$$

where $\gamma \in (0, 1)$ is the step size, $(Z_n)_{n \in \mathbb{N}}$ are i.i.d. standard d -dimensional Gaussian random variables, for all $x \in \mathbb{R}^d$,

$$\mu_\gamma(x) = -\nabla U_\gamma(x) + \frac{\gamma}{2} \left((\nabla^2 U \nabla U)_\gamma(x) - \vec{\Delta}(\nabla U)_\gamma(x) \right),$$

and $\sigma_\gamma(x) = \text{diag} \left((\sigma_\gamma^{(k)}(x))_{k \in \{1, \dots, d\}} \right)$ with

$$\sigma_\gamma^{(k)}(x) = \sqrt{2 + \frac{2\gamma^2}{3} \sum_{j=1}^d |\nabla^2 U_\gamma^{(k,j)}(x)|^2 - 2\gamma \nabla^2 U_\gamma^{(k,k)}(x)}.$$

Note that, for all $x \in \mathbb{R}^d$ and $k = 1, \dots, d$, the term $|\sigma_\gamma^{(k)}(x)| \geq \sqrt{2/3}$, which implies the covariance matrix $\sigma_\gamma(x)$ is positive-definite. The Markov kernel R_γ associated with (2) is given by

$$R_\gamma(x, A) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathbf{1}_A(x + \mu_\gamma(x)\gamma + \sigma_\gamma(x)\sqrt{\gamma}z) e^{-|z|^2/2} dz,$$

for all $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Remark 4. *The tamed order 1.5 scheme (2) is obtained using the Itô-Taylor (known also as Wagner-Platen) expansion, which can be written explicitly as follows:*

$$\begin{aligned} X_{n+1} = X_n - \nabla U_\gamma(X_n)\gamma + \frac{\gamma^2}{2} \left((\nabla^2 U \nabla U)_\gamma(X_n) - \vec{\Delta}(\nabla U)_\gamma(X_n) \right) \\ + \sqrt{2\gamma} \bar{Z}_{n+1} - \sqrt{2} \nabla^2 U_\gamma(X_n) \tilde{Z}_{n+1} \end{aligned} \quad (3)$$

where $(\bar{Z}_n)_{n \in \mathbb{N}}$ are i.i.d. standard d -dimensional Gaussian random variables, and $(\tilde{Z}_n)_{n \in \mathbb{N}}$ are i.i.d. d -dimensional Gaussian random variables with mean $\mathbf{0}$ and covariance $\frac{1}{3}\gamma^3 \mathbf{I}_d$ defined by

$$\tilde{Z}_{n+1} = \int_{t_n}^{t_{n+1}} \int_{t_n}^s dw_r ds.$$

Meanwhile, for any $n \in \mathbb{N}$, $k, l = 1, \dots, d$, the covariance between \bar{Z}_{n+1} and \tilde{Z}_{n+1} is given by

$$\mathbb{E} \left(\sqrt{\gamma} \bar{Z}_{n+1}^{(k)} \tilde{Z}_{n+1}^{(l)} \right) = \begin{cases} \frac{1}{2}\gamma^2, & \text{for } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

One observes that the scheme (3) is Markovian, and $\text{Law}(X_n)$ is the same as $\text{Law}(\bar{X}_n)$, for any $n \in \mathbb{N}$.

Denote by $(P_t)_{t \geq 0}$ the semigroup associated with (1). For all $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, we have $P_t(x, A) = \mathbb{E}[\mathbf{1}_{\{x_t \in A\}} | x_0 = x]$. In addition, for all $x \in \mathbb{R}^d$ and $h \in C^2(\mathbb{R}^d)$, the infinitesimal generator \mathcal{A} associated with (1) is defined by

$$\mathcal{A}h(x) = -\nabla U(x) \nabla h(x) + \Delta h(x).$$

For any $a > 0$, define the Lyapunov function $V_a : \mathbb{R}^d \rightarrow [1, +\infty)$ for all $x \in \mathbb{R}^d$ by

$$V_a(x) = \exp \left(a(1 + |x|^2)^{1/2} \right).$$

Then, for the local Lipschitz drift, one obtains the following convergence results.

Theorem 1. Assume **H1**, **H2** and **H3** are satisfied. Then, there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \mathbb{R}^d$, $\gamma \in (0, 1)$ and $n \in \mathbb{N}$,

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq C(\lambda^{n\gamma} V_c(x) + \gamma^{2+\beta}), \quad (4)$$

where c is given in (13) and for all $\gamma \in (0, 1)$,

$$W_2^2(\pi_\gamma, \pi) \leq C\gamma^{2+\beta}.$$

Theorem 2. Assume **H1** and **H2** are satisfied. There exist $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \mathbb{R}^d$, $\gamma \in (0, 1)$ and $n \in \mathbb{N}$,

$$\|\delta_x R_\gamma^n - \pi\|_{V_c^{1/2}} \leq C(\lambda^{n\gamma} V_c(x) + \gamma), \quad (5)$$

where c is given in (13) and for all $\gamma \in (0, 1)$,

$$\|\pi_\gamma - \pi\|_{V_c^{1/2}} \leq C\gamma. \quad (6)$$

Moreover, the Lipschitz case is considered in order to provide explicit constants for the moment bounds and the convergence in Wasserstein distance. More precisely, the drift coefficient $\nabla U(x)$ is assumed to be Lipschitz continuous and **H2** is replaced by the following assumptions:

H4 There exists $L_1 > 0$, such that for all $x, y \in \mathbb{R}^d$, $|\nabla U(x) - \nabla U(y)| \leq L_1|x - y|$.

H5 There exists $L_2 > 0$, such that for all $x, y \in \mathbb{R}^d$, $|\nabla^2 U(x) - \nabla^2 U(y)| \leq L_2|x - y|$.

H6 There exists $L > 0$, such that for all $x, y \in \mathbb{R}^d$, $|\nabla^2(\nabla U)(x) - \nabla^2(\nabla U)(y)| \leq L|x - y|$.

Theorem 3. Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2 + 2L_2^2} \wedge \frac{4m}{384L_1^4 + 2L_1^2 + L_2^2 + 14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2} \wedge \frac{1}{L}\right)$. Then, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq e^{-mn\gamma} \left(2|x - x^*|^2 + \frac{2d}{m}\right) + \bar{C}\gamma^3,$$

where the constant \bar{C} depends on d^2 , and the explicit expression is given in the proof.

Corollary 1. Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2 + 2L_2^2} \wedge \frac{4m}{384L_1^4 + 2L_1^2 + L_2^2 + 14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2} \wedge \frac{1}{L}\right)$. If one considers a multivariate Gaussian as the target distribution, then for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq e^{-mn\gamma} \left(2|x - x^*|^2 + \frac{2d}{m}\right) + \tilde{C}\gamma^3.$$

where the constant \tilde{C} depends on d , and the explicit expression is given in the proof.

3 Local Lipschitz case

3.1 Moment bounds

It is a well-known result that by **H1** and **H2**, the SDE (1) has a unique strong solution. One then needs to obtain moment bounds of the SDE (1) and the numerical scheme (2) before considering the convergence results.

By using Foster-Lyapunov conditions, one can obtain the exponential moment bounds for the solution of SDE (1). More concretely, the application of Theorem 1.1, 6.1 in [7] and Theorem 2.2 in [21] yields the following results.

Proposition 1. Assume **H1** and **H2** are satisfied. For all $a > 0$, there exists $b_a > 0$, such that for all $x \in \mathbb{R}^d$,

$$\mathcal{A}V_a(x) \leq -aV_a(x) + ab_a,$$

and

$$\sup_{t \geq 0} P_t V_a(x) \leq V_a(x) + b_a.$$

Moreover, there exist $C_a > 0$ and $\rho_a \in (0, 1)$ such that for all $t > 0$ and probability measures μ_0, ν_0 on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying $\mu_0(V_a) + \nu_0(V_a) < +\infty$,

$$\|\mu_0 P_t - \nu_0 P_t\|_{V_a} \leq C_a \rho_a^t \|\mu_0 - \nu_0\|_{V_a}, \quad \|\mu_0 P_t - \pi\|_{V_a} \leq C_a \rho_a^t \mu_0(V_a).$$

Proof. Please refer to Proposition 1 in [2] for the detailed proof. \square

The proposition below provides a uniform bound for exponential moments of the Markov chain $(\bar{X}_k)_{k \geq 0}$.

Proposition 2. Assume **H1** and **H2** are satisfied. Then, there exist constants $b, c, M > 0$, such that for all $x \in \mathbb{R}^d$ and $\gamma \in (0, 1)$,

$$R_\gamma V_c(x) \leq e^{-\frac{7}{3}c^2\gamma} V_c(x) + \gamma b \mathbf{1}_{\bar{B}(0, M)}(x),$$

and for all $n \in \mathbb{N}$

$$R_\gamma^n V_c(x) \leq e^{-\frac{7}{3}c^2n\gamma} V_c(x) + \frac{3b}{7c^2} e^{\frac{7}{3}c^2\gamma}.$$

Moreover, this guarantees that the Gaussian kernel R_γ has a unique invariant measure π_γ and R_γ is geometrically ergodic w.r.t. π_γ .

Proof. We use the scheme (2) throughout the proof. First, one observes that by **H1**, for $\gamma \in (0, 1)$, the following holds

$$\liminf_{|x| \rightarrow +\infty} \frac{x}{|x|} \nabla U_\gamma(x) - \frac{\gamma}{2|x|} |\nabla U_\gamma(x)|^2 > 0. \quad (7)$$

Indeed, by **H1**, there exist $M', \kappa > 0$ such that for all $|x| \geq M'$, $x \in \mathbb{R}^d$, $x \nabla U(x) \geq \kappa |x| |\nabla U(x)|$. Then, we have for all $|x| \geq M'$, $x \in \mathbb{R}^d$,

$$\begin{aligned} & \frac{x}{|x|} \nabla U_\gamma(x) - \frac{\gamma}{2|x|} |\nabla U_\gamma(x)|^2 \\ & \geq \frac{1}{2|x|(1 + \gamma^{3/2} |\nabla U(x)|^{3/2})^{2/3}} \left(2\kappa |x| |\nabla U(x)| - \frac{\gamma |\nabla U(x)|^2}{(1 + \gamma^{3/2} |\nabla U(x)|^{3/2})^{2/3}} \right) \\ & \geq \frac{|\nabla U(x)|}{2|x|(1 + \gamma^{3/2} |\nabla U(x)|^{3/2})^{2/3}} \left(2\kappa |x| - \frac{\sqrt[3]{2}\gamma |\nabla U(x)|}{1 + \gamma |\nabla U(x)|} \right) \\ & \geq \frac{|\nabla U(x)|}{2(1 + \gamma^{3/2} |\nabla U(x)|^{3/2})^{2/3}} \left(2\kappa - \frac{\sqrt[3]{2}}{|x|} \right). \end{aligned}$$

Meanwhile, by **H1**, there exist $M'', K > 0$ such that for all $|x| \geq M''$, $x \in \mathbb{R}^d$, $|\nabla U| \geq K$. Note that $f(x) = x/(1 + x^{3/2})^{2/3}$ is a non-decreasing function for all $x \geq 0$. Then, one obtains (7), since for all $x \in \mathbb{R}^d$, $|x| \geq \max(M', M'', \sqrt[3]{2}\kappa^{-1})$

$$\frac{x}{|x|} \nabla U_\gamma(x) - \frac{\gamma}{2|x|} |\nabla U_\gamma(x)|^2 \geq \frac{\kappa K}{2(1 + \gamma^{3/2} K^{3/2})^{2/3}}.$$

The function $f(x) = (1 + |x|^2)^{1/2}$ is Lipschitz continuous with Lipschitz constant equal to 1. Let $\bar{X}_0 = x$, then for all $x \in \mathbb{R}^d$, applying log Sobolev inequality (see Proposition 5.5.1 in [5] and Appendix A for a detailed proof) gives,

$$R_\gamma V_a(x) = \mathbb{E}_x(V_a(\bar{X}_1)) \leq e^{\frac{7}{3}\gamma a^2} \exp \left\{ a \mathbb{E}((1 + |\bar{X}_1|^2)^{1/2} | \bar{X}_0 = x) \right\}, \quad (8)$$

which using Jensen's inequality yields

$$R_\gamma V_a(x) \leq e^{\frac{7}{3}\gamma a^2} \exp \left\{ a \left(1 + \mathbb{E} \left(|\bar{X}_0 + \mu_\gamma(\bar{X}_0)\gamma + \sigma_\gamma(\bar{X}_0)\sqrt{\gamma}Z_1|^2 \mid \bar{X}_0 = x \right) \right)^{1/2} \right\}. \quad (9)$$

Note that we have

$$|\sigma_\gamma(\bar{X}_0)\sqrt{\gamma}Z_1|^2 = \sum_{i=1}^d \left(2\gamma + \frac{2\gamma^3}{3} \sum_{j=1}^d \left| \nabla^2 U_\gamma^{(i,j)}(\bar{X}_0) \right|^2 - 2\gamma^2 \nabla^2 U_\gamma^{(i,i)}(\bar{X}_0) \right) \left| Z_1^{(i)} \right|^2.$$

By taking the conditional expectation on both sides, the above equation becomes

$$\mathbb{E} \left[|\sigma_\gamma(\bar{X}_0)\sqrt{\gamma}Z_1|^2 \mid \bar{X}_0 = x \right] \leq 2\gamma d + \frac{2\gamma^3}{3} |\nabla^2 U_\gamma(x)|^2 - 2\gamma^2 \text{tr}(\nabla^2 U_\gamma(x)) \leq \frac{14}{3}d\gamma. \quad (10)$$

Then, by inserting (10) into (9), one obtains

$$R_\gamma V_a(x) \leq e^{\frac{7}{3}\gamma a^2} \exp \left\{ a \left(1 + A_\gamma(x) + \frac{14}{3}d\gamma \right)^{1/2} \right\}, \quad (11)$$

where

$$A_\gamma(x) = \left| x - \nabla U_\gamma(x)\gamma + \frac{\gamma^2}{2} \left((\nabla^2 U \nabla U)_\gamma(x) - \vec{\Delta}(\nabla U)_\gamma(x) \right) \right|^2.$$

Then, expanding the square yields

$$\begin{aligned} A_\gamma(x) &= |x|^2 - 2\gamma x \nabla U_\gamma(x) + \gamma^2 |\nabla U_\gamma(x)|^2 - \gamma^2 x \vec{\Delta}(\nabla U)_\gamma(x) + \frac{\gamma^4}{4} \left| \vec{\Delta}(\nabla U)_\gamma(x) \right|^2 \\ &\quad + \gamma^2 x (\nabla^2 U \nabla U)_\gamma(x) - \gamma^3 \nabla U_\gamma(x) (\nabla^2 U \nabla U)_\gamma(x) + \gamma^3 \nabla U_\gamma(x) \vec{\Delta}(\nabla U)_\gamma(x) \\ &\quad + \frac{\gamma^4}{4} \left| (\nabla^2 U \nabla U)_\gamma(x) \right|^2 - \frac{\gamma^4}{2} (\nabla^2 U \nabla U)_\gamma(x) \vec{\Delta}(\nabla U)_\gamma(x). \end{aligned}$$

By (7), there exist $M_1, \kappa_1 > 0$ such that for all $|x| \geq M_1$,

$$x \nabla U_\gamma(x) - \frac{\gamma}{2} |\nabla U_\gamma(x)|^2 > \kappa_1 |x|.$$

Thus, by using Remark 3, for all $|x| \geq \max\{1, M_1\}$,

$$\begin{aligned} A_\gamma(x) + \frac{14}{3}d\gamma &\leq |x|^2 - 2\gamma\kappa_1|x| + \gamma^{3/2} + \frac{1}{4}\gamma^3 + 3\gamma + 2\gamma^{3/2} + \frac{1}{4}\gamma^2 + \frac{1}{2}\gamma^{5/2} + \frac{14}{3}d\gamma \\ &\leq |x|^2 - 2\gamma\kappa_1|x| + \frac{35}{3}d\gamma. \end{aligned}$$

Denote by $M = \max\{1, M_1, \frac{35}{3}d(\kappa_1)^{-1}\}$, for all $x \in \mathbb{R}^d$, $|x| \geq M$,

$$A_\gamma(x) + \frac{14}{3}d\gamma \leq |x|^2 - \gamma\kappa_1|x|.$$

For $t \in [0, 1]$, $(1-t)^{1/2} \leq 1-t/2$ and $g(x) = x/(1+x^2)^{1/2}$ is a non-decreasing function for all $x \geq 0$. Then, for all $x \in \mathbb{R}^d$, $|x| \geq M$

$$\begin{aligned} \left(1 + A_\gamma(x) + \frac{14}{3}d\gamma\right)^{1/2} &\leq (1 + |x|^2)^{1/2} \left(1 - \frac{7\gamma}{3} \frac{3\kappa_1|x|}{7(1+|x|^2)}\right)^{1/2} \\ &\leq (1 + |x|^2)^{1/2} - \frac{7\gamma}{3} \frac{3\kappa_1 M}{14(1+M^2)^{1/2}}. \end{aligned} \quad (12)$$

By substituting (12) into (11) and completing the square, one obtains, for $|x| \geq M$,

$$R_\gamma V_c(x) \leq e^{-\frac{7}{3}c^2\gamma} V_c(x),$$

where

$$c = \frac{3\kappa_1 M}{28(1+M^2)^{1/2}}. \quad (13)$$

For the case $|x| \leq M$, by **H2**, the following result can be obtained:

$$A_\gamma(x) \leq |x|^2 + c_3\gamma(1+M)^{4\rho+2},$$

where c_3 is a positive constant (that depends on d and K). Then, by using $(1+s_1+s_2)^{1/2} \leq (1+s_1)^{1/2} + s_2/2$ for $s_1, s_2 \geq 0$,

$$\left(1 + A_\gamma(x) + \frac{14}{3}d\gamma\right)^{1/2} \leq (1 + |x|^2)^{1/2} + \gamma \left(\frac{c_3}{2}(1+M)^{4\rho+2} + \frac{7d}{3}\right).$$

Thus,

$$R_\gamma V_c(x) \leq e^{\theta\gamma} V_c(x),$$

where $\theta = \frac{7}{3}c^2 + c(\frac{c_3}{2}(1+M)^{4\rho+2} + \frac{7}{3}d)$. Moreover, for $|x| \leq M$,

$$R_\gamma V_c(x) - e^{-\frac{7}{3}c^2\gamma} V_c(x) \leq e^{\theta\gamma} (1 - e^{-\gamma(\frac{7}{3}c^2 + \theta)}) V_c(x) \leq \gamma e^{\theta\gamma} \left(\frac{7}{3}c^2 + \theta\right) V_c(x).$$

Denote by $b = e^{(\theta\gamma + c\sqrt{1+M^2})} (\frac{7}{3}c^2 + \theta)$, one obtains

$$R_\gamma V_c(x) \leq e^{-\frac{7}{3}c^2\gamma} V_c(x) + \gamma b \mathbf{1}_{\overline{B}(0,M)}(x).$$

Then by induction, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\begin{aligned} R_\gamma^n V_c(x) &\leq e^{-\frac{7}{3}c^2n\gamma} V_c(x) + \frac{1 - e^{-\frac{7}{3}c^2n\gamma}}{1 - e^{-\frac{7}{3}c^2\gamma}} \gamma b \\ &\leq e^{-\frac{7}{3}c^2n\gamma} V_c(x) + \frac{3b}{7c^2} e^{\frac{7}{3}c^2\gamma}, \end{aligned}$$

the last inequality holds since $1 - e^{-\frac{7}{3}c^2\gamma} = \int_0^\gamma \frac{7}{3}c^2 e^{-\frac{7}{3}c^2s} ds \geq \frac{7}{3}c^2\gamma e^{-\frac{7}{3}c^2\gamma}$. Finally, since any compact set on \mathbb{R}^d is accessible and small for R_γ , then by section 3.1 in [7] and Theorem 15.0.1 in [8], for all $\gamma \in (0, 1)$, R_γ has a unique invariant measure π_γ and it is geometrically ergodic w.r.t. π_γ . \square

The results in Proposition 1 and 2 provide exponential moment bounds for the solution of SDE (1) and the scheme (2), which enable us to consider the total variation and Wasserstein distance between the target distribution π and the n -th iteration of the MCMC algorithm.

3.2 Proof of Theorem 1

In order to obtain the Wasserstein distance, the assumption **H3** is needed, which assumes the convexity of U . We consider the linear interpolation of the scheme (3) given by

$$\bar{x}_t = \bar{x}_0 - \int_0^t \nabla \tilde{U}_\gamma(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) ds + \sqrt{2} w_t, \quad (14)$$

for all $t \geq 0$, where

$$\nabla \tilde{U}_\gamma(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = \nabla U_\gamma(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) + \nabla U_{1,\gamma}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) + \nabla U_{2,\gamma}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}),$$

with

$$\nabla U_{1,\gamma}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = - \int_{\lfloor s/\gamma \rfloor \gamma}^s \left((\nabla^2 U \nabla U)_\gamma(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) - \vec{\Delta}(\nabla U)_\gamma(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) \right) dr,$$

$$\nabla U_{2,\gamma}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = \sqrt{2} \int_{\lfloor s/\gamma \rfloor \gamma}^s \nabla^2 U_\gamma(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) dw_r.$$

Note that the linear interpolation (14) and the scheme (3) coincide at grid points, i.e. for any $n \in \mathbb{N}$, $X_n = \bar{x}_{n\gamma}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration associated with $(w_t)_{t \geq 0}$. For any $n \in \mathbb{N}$, denote by $\mathbb{E}^{\mathcal{F}_{n\gamma}}[\cdot]$ the expectation conditional on $\mathcal{F}_{n\gamma}$.

Lemma 1. *Assume **H1** and **H2** are satisfied. Then, there exists a constant $C > 0$ such that for all $p > 0$, $\gamma \in (0, 1)$, $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,*

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_{1,\gamma}(t, \bar{x}_{n\gamma})|^p] \leq C \gamma^p V_c(\bar{x}_{n\gamma}),$$

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_{2,\gamma}(t, \bar{x}_{n\gamma})|^p] \leq C \gamma^{\frac{p}{2}} V_c(\bar{x}_{n\gamma}).$$

Proof. Consider a polynomial function $f(|x|) \in C_{poly}(\mathbb{R}_+, \mathbb{R}_+)$, then there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$, $f(|x|) \leq C V_c(x)$. For $p > 1$, by applying Hölder's inequality and Remark 2, one obtains

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_{1,\gamma}(t, \bar{x}_{n\gamma})|^p] &= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| - \int_{n\gamma}^t \left((\nabla^2 U \nabla U)_\gamma(\bar{x}_{n\gamma}) - \vec{\Delta}(\nabla U)_\gamma(\bar{x}_{n\gamma}) \right) dr \right|^p \right] \\ &\leq C \gamma^{p-1} \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|(\nabla^2 U \nabla U)_\gamma(\bar{x}_{n\gamma})|^p] dr \\ &\quad + C \gamma^{p-1} \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\vec{\Delta}(\nabla U)_\gamma(\bar{x}_{n\gamma})|^p] dr \\ &\leq C \gamma^p V(\bar{x}_{n\gamma}), \end{aligned}$$

The second inequality can be proved using similar arguments. For the case $0 < p \leq 1$, Jensen's inequality is used to obtain the desired result. \square

Lemma 2. *Assume **H1** and **H2** are satisfied. Then, there exists a constant $C > 0$ such that for all $p > 0$, $\gamma \in (0, 1)$, $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,*

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_t - \bar{x}_{n\gamma}|^p] \leq C \gamma^{\frac{p}{2}} V_c(\bar{x}_{n\gamma}),$$

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|x_t - x_{n\gamma}|^p] \leq C \gamma^{\frac{p}{2}} V_c(x_{n\gamma}).$$

Proof. For $p > 1$, by using Hölder's inequality, Remark 2 and Lemma 1, we have

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_t - \bar{x}_{n\gamma}|^p] &= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| -\int_{n\gamma}^t \nabla \tilde{U}_\gamma(s, \bar{x}_{n\gamma}) ds + \sqrt{2} \int_{n\gamma}^t dw_s \right|^p \right] \\ &\leq C\gamma^{p-1} \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_\gamma(\bar{x}_{n\gamma}) + \nabla U_{1,\gamma}(s, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(s, \bar{x}_{n\gamma})|^p] ds + C\gamma^{\frac{p}{2}} \\ &\leq C\gamma^{\frac{p}{2}} V_c(\bar{x}_{n\gamma}).\end{aligned}$$

For the case $0 < p \leq 1$, one can use Jensen's inequality to obtain

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_t - \bar{x}_{n\gamma}|^p] &\leq \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left| \int_{n\gamma}^t \nabla \tilde{U}_\gamma(s, \bar{x}_{n\gamma}) ds + \sqrt{2} \int_{n\gamma}^t dw_s \right|^p \right)^{\frac{1}{p}} \\ &\leq C\gamma^{\frac{p}{2}} V_c(\bar{x}_{n\gamma}),\end{aligned}$$

Similarly, for $p > 1$, by using Hölder's inequality, one obtains

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_{n\gamma}} [|x_t - x_{n\gamma}|^p] &= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| -\int_{n\gamma}^t \nabla U(x_s) ds + \sqrt{2} \int_{n\gamma}^t dw_s \right|^p \right] \\ &\leq C\gamma^{p-1} \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left(1 + |x_s|^{p(\rho+\beta)} \right) ds + C\gamma^{\frac{p}{2}} \\ &\leq C\gamma^{\frac{p}{2}} V_c(x_{n\gamma}),\end{aligned}$$

where the last inequality holds due to Proposition 1. The case $p \in (0, 1]$ follows from the application of Jensen's inequality. \square

Lemma 3. Assume **H1** and **H2** are satisfied. Then, there exists a constant $C > 0$ such that for all $\gamma \in (0, 1)$, $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma})|^2] \leq C\gamma^2 V_c(\bar{x}_{n\gamma}).$$

Proof. For any $t \in [n\gamma, (n+1)\gamma)$, applying Itô's formula to $\nabla U(\bar{x}_t)$ gives, almost surely

$$\begin{aligned}&\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) \\ &= -\int_{n\gamma}^t \left(\nabla^2 U(\bar{x}_r) \nabla \tilde{U}_\gamma(r, \bar{x}_{n\gamma}) - \vec{\Delta}(\nabla U)(\bar{x}_r) \right) dr + \sqrt{2} \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) dw_r \\ &= -\int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma}) dr - \int_{n\gamma}^t \nabla^2 U(\bar{x}_{n\gamma}) \nabla U_\gamma(\bar{x}_{n\gamma}) dr \\ &\quad - \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr \\ &\quad + \sqrt{2} \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r + \sqrt{2} \int_{n\gamma}^t \nabla^2 U(\bar{x}_{n\gamma}) dw_r \\ &\quad + \int_{n\gamma}^t \left(\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) dr + \int_{n\gamma}^t \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) dr.\end{aligned}$$

By further calculations, one obtains

$$\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma}) \leq \sum_{i=1}^5 G_i(t), \quad (15)$$

where

$$\begin{aligned}
G_1(t) &= - \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma}) dr, \\
G_2(t) &= - \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr, \\
G_3(t) &= \sqrt{2} \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r, \\
G_4(t) &= \int_{n\gamma}^t \left(\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) dr, \\
G_5(t) &= |\nabla^2 U(\bar{x}_{n\gamma})| |\nabla U(\bar{x}_{n\gamma})|^2 \gamma^2 + |\bar{x}_{n\gamma}| |\nabla^2 U(\bar{x}_{n\gamma})|^2 |\nabla U(\bar{x}_{n\gamma})|^2 \gamma^2 \\
&\quad + \gamma^{3/2} |\bar{x}_{n\gamma}| |\vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})|^2 + \sqrt{2} \gamma |\nabla^2 U(\bar{x}_{n\gamma})|^2 (w_t - w_{n\gamma})
\end{aligned}$$

Then, by squaring both sides of (15) and taking conditional expectation yields

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma})|^2 \right] \leq C \sum_{i=1}^5 \mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_i(t)|^2].$$

By using Cauchy-Schwarz inequality, Proposition 2, Remark 2 and Lemma 2, one obtains

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_1(t)|^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|(\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma})|^2] dr \\
&\leq C \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_r| + |\bar{x}_{n\gamma}|)^{4\rho-4+4\beta} |\bar{x}_r - \bar{x}_{n\gamma}|^2 \right] dr \\
&\leq C \gamma \int_{n\gamma}^t \sqrt{\mathbb{E}^{\mathcal{F}_{n\gamma}} [V_c(\bar{x}_r) + V_c(\bar{x}_{n\gamma})]} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_r - \bar{x}_{n\gamma}|^4] dr \\
&\leq C \gamma^3 V_c(\bar{x}_{n\gamma}).
\end{aligned} \tag{16}$$

Similarly, by Cauchy-Schwarz inequality, Proposition 2 and Remark 2, one obtains

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_2(t)|^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma}))|^2] dr \\
&\leq C \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_r|)^{2\rho-2+2\beta} |\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})|^2 \right] dr \\
&\leq C \gamma \int_{n\gamma}^t \sqrt{\mathbb{E}^{\mathcal{F}_{n\gamma}} [V_c(\bar{x}_r)]} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma})|^4 + |\nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})|^4] dr \\
&\leq C \gamma^3 V_c(\bar{x}_{n\gamma}),
\end{aligned} \tag{17}$$

where the last inequality is obtained by applying Lemma 1. Moreover, applying Cauchy-Schwarz inequality, Proposition 2, Lemma 2 and Remark 2 yields

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_3(t)|^2] &\leq C \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})|^2] dr \\
&\leq C \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_r| + |\bar{x}_{n\gamma}|)^{2\rho-4+2\beta} |\bar{x}_r - \bar{x}_{n\gamma}|^2 \right] dr \\
&\leq C \int_{n\gamma}^t \sqrt{\mathbb{E}^{\mathcal{F}_{n\gamma}} [V_c(\bar{x}_r) + V_c(\bar{x}_{n\gamma})]} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_r - \bar{x}_{n\gamma}|^4] dr \\
&\leq C \gamma^2 V_c(\bar{x}_{n\gamma}).
\end{aligned}$$

Furthermore, one obtains by using Cauchy-Schwarz inequality, Proposition 2, Lemma 2 and **H2** that

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_4(t)|^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right|^2 \right] dr \\
&\leq C\gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_r| + |\bar{x}_{n\gamma}|)^{2\rho-4} |\bar{x}_r - \bar{x}_{n\gamma}|^{2\beta} \right] dr \\
&\leq C\gamma \int_{n\gamma}^t \sqrt{\mathbb{E}^{\mathcal{F}_{n\gamma}} [V_c(\bar{x}_r) + V_c(\bar{x}_{n\gamma})]} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_r - \bar{x}_{n\gamma}|^{4\beta}] dr \\
&\leq C\gamma^{2+\beta} V_c(\bar{x}_{n\gamma}).
\end{aligned} \tag{18}$$

The estimate of $\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_5(t)|^2]$ can be obtained by straightforward calculations, and we have $\mathbb{E}^{\mathcal{F}_{n\gamma}} [|G_5(t)|^2] \leq C\gamma^3 V_c(\bar{x}_{n\gamma})$. Therefore,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma})|^2] \leq C\gamma^2 V_c(\bar{x}_{n\gamma}).$$

□

For any $x, \bar{x} \in \mathbb{R}^d$, denote by $M(x, \bar{x})$ a matrix whose (i, j) -th entry is $\sum_{k=1}^d \frac{\partial^3 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (x^{(k)} - \bar{x}^{(k)})$. One then obtains the following results.

Lemma 4. Assume **H2** holds. Then, there exists a constant $C > 0$ such that for any $x, \bar{x} \in \mathbb{R}^d$, and $i = 1, \dots, d$,

$$|\nabla^2 U(x) - \nabla^2 U(\bar{x}) - M(x, \bar{x})| \leq L(1 + |x| + |\bar{x}|)^{\rho-2} |x - \bar{x}|^{1+\beta}.$$

Proof. By the mean value theorem, for all $x, \bar{x} \in \mathbb{R}^d$, $i, j = 1, \dots, d$, there exists $q \in [0, 1]$, such that

$$\frac{\partial^2 U(x)}{\partial x^{(i)} \partial x^{(j)}} - \frac{\partial^2 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)}} = \sum_{k=1}^d \frac{\partial^3 U(qx + (1-q)\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (x^{(k)} - \bar{x}^{(k)}).$$

Then by Cauchy-Schwarz inequality and **H2**

$$\begin{aligned}
&|\nabla^2 U(x) - \nabla^2 U(\bar{x}) - M(x, \bar{x})| \\
&= \left(\sum_{i,j=1}^d \left| \frac{\partial^2 U(x)}{\partial x^{(i)} \partial x^{(j)}} - \frac{\partial^2 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)}} - M^{(i,j)}(x, \bar{x}) \right|^2 \right)^{1/2} \\
&= \left(\sum_{i,j=1}^d \left| \sum_{k=1}^d \frac{\partial^3 U(qx + (1-q)\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (x^{(k)} - \bar{x}^{(k)}) - \sum_{k=1}^d \frac{\partial^3 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (x^{(k)} - \bar{x}^{(k)}) \right|^2 \right)^{1/2} \\
&\leq \left(\sum_{i,j=1}^d \sum_{k=1}^d \left| \frac{\partial^3 U(qx + (1-q)\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} - \frac{\partial^3 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \right|^2 |x - \bar{x}|^2 \right)^{1/2} \\
&\leq L(1 + |x| + |\bar{x}|)^{\rho-2} |x - \bar{x}|^{1+\beta}.
\end{aligned}$$

□

Lemma 5. Assume **H1** and **H2** are satisfied. Then, there exists a constant $C > 0$ such that for all $\gamma \in (0, 1)$, $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right|^2 \right] \leq C\gamma^2 V_c(\bar{x}_{n\gamma}).$$

Proof. By using conditional Itô's isometry and Lemma 2, one obtains

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right|^2 \right] &\leq C \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t |M(\bar{x}_r, \bar{x}_{n\gamma})|^2 dr \right] \\
&= C \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \left(\sum_{i,j=1}^d \left| \sum_{k=1}^d \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (\bar{x}_r^{(k)} - \bar{x}_{n\gamma}^{(k)}) \right|^2 \right) dr \right] \\
&\leq C \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_{n\gamma}|)^{2(\rho-2+\beta)} |\bar{x}_r - \bar{x}_{n\gamma}|^2 \right] dr \\
&\leq C \gamma^2 V_c(\bar{x}_{n\gamma}).
\end{aligned}$$

□

Lemma 6. Assume **H1** and **H2** are satisfied. Then, there exists a constant $C > 0$ such that for all $\gamma \in (0, 1)$, $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \leq C \gamma^3 (V_c(\bar{x}_{n\gamma}) + V_c(x_{n\gamma})).$$

Proof. For any $t \in [n\gamma, (n+1)\gamma)$, one observes that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \left\{ \nabla U(x_r) - \nabla U(x_{n\gamma}) - (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})) - \sqrt{2} \int_{n\gamma}^r \nabla^2 U(x_{n\gamma}) dw_s \right. \right. \\
&\quad \left. \left. + \sqrt{2} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s \right\} dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&\quad + \sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \int_{n\gamma}^r (\nabla^2 U(x_{n\gamma}) - \nabla^2 U(\bar{x}_{n\gamma})) dw_s dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right].
\end{aligned} \tag{19}$$

The second term in (19) can be rewritten as

$$\begin{aligned}
&\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \int_{n\gamma}^r (\nabla^2 U(x_{n\gamma}) - \nabla^2 U(\bar{x}_{n\gamma})) dw_s dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&= \sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\sum_{i=1}^d \int_{n\gamma}^t \sum_{l=1}^d \int_{n\gamma}^r (\nabla^2 U^{(i,l)}(x_{n\gamma}) - \nabla^2 U^{(i,l)}(\bar{x}_{n\gamma})) dw_s^{(l)} dr \right. \\
&\quad \left. \times \sum_{j=1}^d \int_{n\gamma}^t \sum_{k=1}^d \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \left(- \int_{n\gamma}^r \nabla \tilde{U}_\gamma^{(k)}(s, \bar{x}_{n\gamma}) ds + \sqrt{2} \int_{n\gamma}^r dw_s^{(k)} \right) dw_r^{(j)} \right] \\
&\leq C \gamma^3 (V_c(x_{n\gamma}) + V_c(\bar{x}_{n\gamma})).
\end{aligned}$$

where the last inequality holds due to Cauchy-Schwarz inequality, Lemma 1, Proposition 1, 2 and the fact that for any $i, l, j, k = 1, \dots, d$

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \int_{n\gamma}^r (\nabla^2 U^{(i,l)}(x_{n\gamma}) - \nabla^2 U^{(i,l)}(\bar{x}_{n\gamma})) dw_s^{(l)} dr \int_{n\gamma}^t \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \int_{n\gamma}^r \sqrt{2} dw_s^{(k)} dw_r^{(j)} \right] = 0.$$

Then, to estimate the first term of (19), one applies Itô's formula to $\nabla U(x_r)$ and $\nabla U(\bar{x}_r)$ to obtain, almost surely

$$\begin{aligned}
& \nabla U(x_r) - \nabla U(x_{n\gamma}) - (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})) - \sqrt{2} \int_{n\gamma}^r \nabla^2 U(x_{n\gamma}) dw_s + \sqrt{2} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s \\
&= - \int_{n\gamma}^r \left(\nabla^2 U(x_s) \nabla U(x_s) - \tilde{\Delta}(\nabla U)(x_s) \right) ds + \sqrt{2} \int_{n\gamma}^r (\nabla^2 U(x_s) - \nabla^2 U(x_{n\gamma})) dw_s \\
&\quad + \int_{n\gamma}^r \left(\nabla^2 U(\bar{x}_s) \nabla \tilde{U}_\gamma(s, \bar{x}_{n\gamma}) - \tilde{\Delta}(\nabla U)(\bar{x}_s) \right) ds - \sqrt{2} \int_{n\gamma}^r (\nabla^2 U(\bar{x}_s) - \nabla^2 U(\bar{x}_{n\gamma})) dw_s.
\end{aligned} \tag{20}$$

By using Cauchy-Schwarz inequality, equation (19) yields

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&\leq \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma \int_{n\gamma}^t |\nabla U(x_r) - \nabla U(x_{n\gamma}) - (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})) \right. \right. \\
&\quad \left. \left. - \sqrt{2} \int_{n\gamma}^r \nabla^2 U(x_{n\gamma}) dw_s + \sqrt{2} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s|^2 dr \right] \right)^{1/2} \\
&\quad + C\gamma^3 (V_c(\bar{x}_{n\gamma}) + V_c(x_{n\gamma})).
\end{aligned}$$

Then, by taking into consideration (20), and by applying Lemma 5, Proposition 1 and 2, one obtains

$$\begin{aligned}
&\leq \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \left| \nabla^2 U(x_s) \nabla U(x_s) - \tilde{\Delta}(\nabla U)(x_s) \right|^2 ds dr \right] \right)^{1/2} \\
&\quad + \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \left| \nabla^2 U(\bar{x}_s) \nabla \tilde{U}_\gamma(s, \bar{x}_{n\gamma}) - \tilde{\Delta}(\nabla U)(\bar{x}_s) \right|^2 ds dr \right] \right)^{1/2} \\
&\quad + \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla^2 U(x_s) - \nabla^2 U(x_{n\gamma})|^2 \right] ds dr \right)^{1/2} \\
&\quad + \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla^2 U(\bar{x}_s) - \nabla^2 U(\bar{x}_{n\gamma})|^2 \right] ds dr \right)^{1/2} \\
&\quad + C\gamma^3 (V_c(\bar{x}_{n\gamma}) + V_c(x_{n\gamma})) \\
&\leq \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |x_s| + |x_{n\gamma}|)^{2\rho-2} |x_s - x_{n\gamma}|^2 \right] ds dr \right)^{1/2} \\
&\quad + \sqrt{C\gamma^2 V_c(\bar{x}_{n\gamma})} \left(\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(1 + |\bar{x}_s| + |\bar{x}_{n\gamma}|)^{2\rho-2} |\bar{x}_s - \bar{x}_{n\gamma}|^2 \right] ds dr \right)^{1/2} \\
&\quad + C\gamma^3 (V_c(\bar{x}_{n\gamma}) + V_c(x_{n\gamma})).
\end{aligned}$$

Finally by using Cauchy-Schwarz inequality and Lemma 2, one obtains

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \leq C\gamma^3 (V_c(\bar{x}_{n\gamma}) + V_c(x_{n\gamma})).$$

□

Proof of Theorem 1. Consider the coupling

$$\begin{cases} x_t = x_0 - \int_0^t \nabla U(x_r) dr + \sqrt{2}w_t, \\ \bar{x}_t = \bar{x}_0 - \int_0^t \nabla \tilde{U}_\gamma(r, \bar{x}_{\lfloor r/\gamma \rfloor \gamma}) dr + \sqrt{2}w_t, \end{cases}$$

where $-\nabla \tilde{U}_\gamma(r, \bar{x}_{\lfloor r/\gamma \rfloor \gamma})$ is defined in (14). Let (x_0, \bar{x}_0) be distributed according to ζ_0 , where $\zeta_0 = \pi \otimes \delta_x$ for all $x \in \mathbb{R}^d$. Define $e_t = x_t - \bar{x}_t$, for all $t \in [n\gamma, (n+1)\gamma)$, $n \in \mathbb{N}$. By Itô's formula, one obtains, almost surely,

$$|e_t|^2 = |e_{n\gamma}|^2 - 2 \int_{n\gamma}^t e_s (\nabla U(x_s) - \nabla \tilde{U}_\gamma(s, \bar{x}_{n\gamma})) ds.$$

Then, taking the expectation conditional on $\mathcal{F}_{n\gamma}$ and taking the derivative on both sides yield

$$\begin{aligned} \frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] &= -2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (\nabla U(x_t) - \nabla \tilde{U}_\gamma(t, \bar{x}_{n\gamma}))] \\ &= 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-\nabla U(x_t) - \nabla U(\bar{x}_t))] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma}))] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-\nabla U(\bar{x}_{n\gamma}) - \nabla U_\gamma(\bar{x}_{n\gamma}))], \end{aligned}$$

which implies by using **H3** and $|a||b| \leq \varepsilon a^2 + (4\varepsilon)^{-1}b^2$, $\varepsilon > 0$,

$$\begin{aligned} \frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] &\leq (2\varepsilon)^{-1} \gamma^3 |\nabla U(\bar{x}_{n\gamma})|^5 - 2(m - \varepsilon) \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma}))]. \end{aligned} \quad (21)$$

By applying Itô's formula to $\nabla U(\bar{x}_t)$, and by calculating $\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(t, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(t, \bar{x}_{n\gamma})$, one obtains (15). Substituting (15) into (21) gives

$$\begin{aligned} \frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] &\leq (2\varepsilon)^{-1} \gamma^3 |\nabla U(\bar{x}_{n\gamma})|^5 - 2(m - \varepsilon) \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma}) dr \right| \right] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr \right| \right] \\ &\quad + 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left(- \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r \right) \right] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t (\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})) dr \right| \right] \\ &\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| (|\nabla^2 U(\bar{x}_{n\gamma})| |\nabla U(\bar{x}_{n\gamma})|^2 \gamma^2 + |\bar{x}_{n\gamma}| |\nabla^2 U(\bar{x}_{n\gamma})|^2 |\nabla U(\bar{x}_{n\gamma})|^2 \gamma^2 \right. \\ &\quad \left. + \gamma^{3/2} |\bar{x}_{n\gamma}| |\vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})|^2 + \sqrt{2} \gamma |\nabla^2 U(\bar{x}_{n\gamma})|^2 (w_t - w_{n\gamma})) \right]. \end{aligned}$$

By Young's inequality and Cauchy-Schwarz inequality,

$$\frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq J_1(t) + J_2(t), \quad (22)$$

where

$$J_1(t) = 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left(- \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r \right) \right],$$

and

$$\begin{aligned}
J_2(t) &= (2\varepsilon)^{-1} \gamma^3 |\nabla U(\bar{x}_{n\gamma})|^5 - 2(m - 5\varepsilon) \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\
&\quad + (2\varepsilon)^{-1} \gamma \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t |(\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma})|^2 dr \right] \\
&\quad + (2\varepsilon)^{-1} \gamma \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t |\nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma}))|^2 dr \right] \\
&\quad + (2\varepsilon)^{-1} \gamma \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t |\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})|^2 dr \right] \\
&\quad + 4(2\varepsilon)^{-1} \gamma^3 (|\nabla^2 U(\bar{x}_{n\gamma})|^2 |\nabla U(\bar{x}_{n\gamma})|^4 + |\bar{x}_{n\gamma}|^2 |\nabla^2 U(\bar{x}_{n\gamma})|^4 |\nabla U(\bar{x}_{n\gamma})|^4 \\
&\quad \quad + |\bar{x}_{n\gamma}|^2 |\vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})|^4 + 2|\nabla^2 U(\bar{x}_{n\gamma})|^4).
\end{aligned}$$

By taking $\varepsilon = \frac{m}{12}$, and by using the results form (16)-(18) in Lemma 3, one obtains

$$J_2(t) \leq C\gamma^{2+\beta} V_c(\bar{x}_{n\gamma}) - \frac{7}{6} m \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2], \quad (23)$$

where $\beta = (0, 1]$. Moreover, one can rewrite $J_1(t)$ as follows

$$\begin{aligned}
J_1(t) &= -2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_t \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma}) - M(\bar{x}_r, \bar{x}_{n\gamma})) dw_r \right] \\
&\quad - 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(e_t - e_{n\gamma}) \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] - 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_{n\gamma} \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right],
\end{aligned}$$

which implies due to Young's inequality, Lemma 4, 2 and the fact that the last term above is zero,

$$\begin{aligned}
J_1(t) &\leq 2\varepsilon \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] + C\gamma^{2+\beta} V_c(\bar{x}_{n\gamma}) \\
&\quad + 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla \tilde{U}_\gamma(r, \bar{x}_{n\gamma})) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right].
\end{aligned}$$

It can be further rewritten as

$$\begin{aligned}
J_1(t) &\leq 2\varepsilon \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] + C\gamma^{2+\beta} V_c(\bar{x}_{n\gamma}) \\
&\quad + 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&\quad + 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) - \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr \right. \\
&\quad \quad \left. \times \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&\quad + 2\sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(\bar{x}_{n\gamma}) - \nabla U_\gamma(\bar{x}_{n\gamma})) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right],
\end{aligned}$$

which, by using Cauchy-Schwarz inequality, Remark 2, Lemma 6, 3 and 5, yields

$$J_1(t) \leq 2\varepsilon \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] + C\gamma^{2+\beta} (V_c(x_{n\gamma}) + V_c(\bar{x}_{n\gamma})) \quad (24)$$

Substituting (24) and (23) into (22) with $\varepsilon = \frac{m}{12}$, one obtains the following result,

$$\frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq -m \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] + C\gamma^{2+\beta} (V_c(x_{n\gamma}) + V_c(\bar{x}_{n\gamma})).$$

The application of Gronwall's lemma yields

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq e^{-m(t-n\gamma)} |e_{n\gamma}|^2 + C\gamma^{3+\beta} (V_c(x_{n\gamma}) + V_c(\bar{x}_{n\gamma})).$$

Finally, by induction, Proposition 1 and 2, one obtains

$$\begin{aligned} \mathbb{E} [|e_{(n+1)\gamma}|^2] &= \mathbb{E} \left[\mathbb{E}^{\mathcal{F}_0} \left[\dots \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_{(n+1)\gamma}|^2] \dots \right] \right] \\ &\leq e^{-m\gamma(n+1)} \mathbb{E} [|e_0|^2] + C\gamma^{3+\beta} \sum_{k=0}^n \mathbb{E} [V_c(\bar{x}_{k\gamma}) + V_c(x_{k\gamma})] e^{-m\gamma(n-k)} \\ &\leq e^{-m\gamma(n+1)} \mathbb{E} [|x_0 - \bar{x}_0|^2] + \frac{3bC}{7c^2m} e^{(\frac{7}{3}c^2+m)\gamma} \gamma^{2+\beta} \\ &\quad + \frac{C}{m} \gamma^{2+\beta} (\mathbb{E} [V_c(x_0)] + b_a) e^{m\gamma} + C\gamma^{3+\beta} \mathbb{E} [V_c(\bar{x}_0)] \sum_{k=0}^n e^{-\frac{7}{3}c^2\gamma k - m\gamma(n-k)}, \end{aligned}$$

where the last inequality holds by using $1 - e^{-m\gamma} \geq m\gamma e^{-m\gamma}$, and this indicates (see Appendix B for a detailed proof)

$$\mathbb{E} [|e_{(n+1)\gamma}|^2] \leq e^{-m\gamma(n+1)} \mathbb{E} [|x_0 - \bar{x}_0|^2] + C\gamma^{2+\beta}, \quad (25)$$

Note that (x_0, \bar{x}_0) is distributed according to ζ_0 , then (4) can be obtained by using Theorem 1 in [11] and the triangle inequality. Furthermore, by Proposition 2 and Corollary 6.11 in [4], as $n \rightarrow +\infty$,

$$\lim_{n \rightarrow +\infty} W_2(\delta_x R_\gamma^n, \pi) = W_2(\pi_\gamma, \pi) = C\gamma^{1+\beta/2},$$

which completes the proof.

3.3 Proof of Theorem 2

By applying the following lemma, one can show that without using **H3**, the rate of convergence in total variation norm is of order 1, which is properly stated in Theorem 2.

Lemma 7. *Assume **H1** and **H2** are satisfied. Let $p \in \mathbb{N}$ and ν_0 be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. There exists $C > 0$ such that for all $\gamma \in (0, 1)$*

$$\text{KL}(\nu_0 R_\gamma^p | \nu_0 P_{p\gamma}) \leq C\gamma^3 \int_{\mathbb{R}^d} \sum_{i=0}^{p-1} \left(\int_{\mathbb{R}^d} V_c(z) R_\gamma^i(y, dz) \right) \nu_0(dy).$$

Proof. Denote by μ_p^y and $\bar{\mu}_p^y$ the laws on $\mathcal{C}([0, p\gamma], \mathbb{R}^d)$ of the SDE (1) and of the linear interpolation (14) of the scheme both started at $y \in \mathbb{R}^d$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration associated with $(w_t)_{t \geq 0}$, and by $(x_t, \bar{x}_t)_{t \geq 0}$ the unique strong solution of

$$\begin{cases} dx_t = -\nabla U(x_t) dt + \sqrt{2} dw_t, \\ d\bar{x}_t = -\nabla \tilde{U}_\gamma(t, \bar{x}_{\lfloor t/\gamma \rfloor \gamma}) dt + \sqrt{2} w_t, \end{cases} \quad (26)$$

where $-\nabla \tilde{U}_\gamma(t, \bar{x}_{\lfloor t/\gamma \rfloor \gamma})$ is defined in (14). Then, by taking into consideration Definition 7 concerning diffusion type processes and Lemma 4.9 which refers to their representations in section 4.2 from [1], Theorem 7.19 in [1] can be applied to obtain the Radon-Nikodym derivative of μ_p^y w.r.t. $\bar{\mu}_p^y$, i.e.

$$\begin{aligned} \frac{d\mu_p^y}{d\bar{\mu}_p^y}((\bar{x}_t)_{t \in [0, p\gamma]}) &= \exp \left(\frac{1}{2} \int_0^{p\gamma} (-\nabla U(\bar{x}_s) + \nabla \tilde{U}_\gamma(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma})) d\bar{x}_s \right. \\ &\quad \left. - \frac{1}{4} \int_0^{p\gamma} (|\nabla U(\bar{x}_s)|^2 - |\nabla \tilde{U}_\gamma(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma})|^2) ds \right). \end{aligned} \quad (27)$$

Note that the assumptions of Theorem 7.19 in [1] are satisfied due to proposition 1 and 2. By using (27), one obtains

$$\begin{aligned}
\text{KL}(\bar{\mu}_p^y | \mu_p^y) &= \mathbb{E}_y \left(-\log \left(\frac{d\mu_p^y}{d\bar{\mu}_p^y}((\bar{x}_t)_{t \in [0, p\gamma]}) \right) \right) \\
&= \frac{1}{4} \int_0^{p\gamma} \mathbb{E}_y \left(\left| \nabla U(\bar{x}_s) - \nabla \tilde{U}_\gamma(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) \right|^2 \right) ds \\
&= \frac{1}{4} \sum_{i=0}^{p-1} \int_{i\gamma}^{(i+1)\gamma} \mathbb{E}_y \left(\left| \nabla U(\bar{x}_s) - \nabla \tilde{U}_\gamma(s, \bar{x}_{i\gamma}) \right|^2 \right) ds \\
&\leq \frac{1}{2} \sum_{i=0}^{p-1} \int_{i\gamma}^{(i+1)\gamma} \mathbb{E}_y \left(\mathbb{E}^{\mathcal{F}_{i\gamma}} \left(\left| \nabla U(\bar{x}_s) - \nabla U(\bar{x}_{i\gamma}) - \nabla U_{1,\gamma}(s, \bar{x}_{i\gamma}) - \nabla U_{2,\gamma}(s, \bar{x}_{i\gamma}) \right|^2 \right) \right) ds \\
&\quad + \frac{1}{2} \sum_{i=0}^{p-1} \int_{i\gamma}^{(i+1)\gamma} \mathbb{E}_y \left(\mathbb{E}^{\mathcal{F}_{i\gamma}} \left(\left| \nabla U(\bar{x}_{i\gamma}) - \nabla U_\gamma(\bar{x}_{i\gamma}) \right|^2 \right) \right) ds \\
&\leq C\gamma^3 \sum_{i=0}^{p-1} \mathbb{E}_y(V_c(\bar{x}_{i\gamma})),
\end{aligned}$$

where the last inequality holds due to Lemma 3. Then, by Theorem 4.1 in [3], it follows that

$$\text{KL}(\delta_y R_\gamma^p | \delta_y P_{p\gamma}) \leq \text{KL}(\bar{\mu}_p^y | \mu_p^y) \leq C\gamma^3 \sum_{i=0}^{p-1} \mathbb{E}_y(V_c(\bar{x}_{i\gamma})).$$

Finally, applying the tower property yields the desired result,

$$\text{KL}(\nu_0 R_\gamma^p | \nu_0 P_{p\gamma}) \leq C\gamma^3 \sum_{i=0}^{p-1} \mathbb{E}(\mathbb{E}_y(V_c(\bar{x}_{i\gamma}))) = C\gamma^3 \int_{\mathbb{R}^d} \sum_{i=0}^{p-1} \left(\int_{\mathbb{R}^d} V_c(z) R_\gamma^i(y, dz) \right) \nu_0(dy).$$

□

Proof of Theorem 2. The proof follows along the same lines as the proof of Theorem 4 in [2], but for the completeness, the details are given below.

By Proposition 1, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
\|\delta_x R_\gamma^n - \pi\|_{V_c^{1/2}} &\leq \|\delta_x P_{n\gamma} - \pi\|_{V_c^{1/2}} + \|\delta_x R_\gamma^n - \delta_x P_{n\gamma}\|_{V_c^{1/2}} \\
&\leq C_{c/2} \rho_{c/2}^{n\gamma} V_c^{1/2}(x) + \|\delta_x R_\gamma^n - \delta_x P_{n\gamma}\|_{V_c^{1/2}}.
\end{aligned}$$

Denote by $k_\gamma = \lceil \gamma^{-1} \rceil$, and by q_γ, r_γ the quotient and the remainder of the Euclidian division of n by k_γ , i.e. $n = q_\gamma k_\gamma + r_\gamma$. Then,

$$\|\delta_x R_\gamma^n - \delta_x P_{n\gamma}\|_{V_c^{1/2}} \leq I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \|\delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma} - \delta_x R_\gamma^n\|_{V_c^{1/2}} \\
I_2 &= \sum_{i=1}^{q_\gamma} \|\delta_x R_\gamma^{(i-1)k_\gamma} P_{(n-(i-1)k_\gamma)\gamma} - \delta_x R_\gamma^{ik_\gamma} P_{(n-ik_\gamma)\gamma}\|_{V_c^{1/2}} \\
&\leq \sum_{i=1}^{q_\gamma} C_{c/2} \rho_{c/2}^{(n-ik_\gamma)\gamma} \|\delta_x R_\gamma^{(i-1)k_\gamma} P_{k_\gamma \gamma} - \delta_x R_\gamma^{ik_\gamma}\|_{V_c^{1/2}}
\end{aligned}$$

By applying Lemma 24 in [12] to I_1 , we have

$$\|\delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma} - \delta_x R_\gamma^n\|_{V_c^{1/2}}^2 \leq 2 \left(\delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma}(V_c) + \delta_x R_\gamma^n(V_c) \right) \times \text{KL}(\delta_x R_\gamma^n | \delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma}). \quad (28)$$

Then, by Proposition 2 and Lemma 7, one obtains

$$\begin{aligned} \text{KL}(\delta_x R_\gamma^n | \delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma}) &\leq C \gamma^3 \sum_{j=0}^{r_\gamma-1} \int_{\mathbb{R}^d} V_c(z) \delta_x R_\gamma^{q_\gamma k_\gamma + j}(dz) \\ &\leq C \gamma^3 (1 + \gamma^{-1}) \left(e^{-\frac{7}{3} c^2 q_\gamma k_\gamma \gamma} V_c(x) + \frac{3b}{7c^2} e^{\frac{7}{3} c^2 \gamma} \right), \end{aligned} \quad (29)$$

where the last inequality holds since $r_\gamma \leq k_\gamma \leq 1 + \gamma^{-1}$. Furthermore, by Proposition 1 and Proposition 2,

$$\delta_x R_\gamma^{q_\gamma k_\gamma} P_{r_\gamma \gamma}(V_c) + \delta_x R_\gamma^n(V_c) \leq 2 \left(e^{-\frac{7}{3} c^2 q_\gamma k_\gamma \gamma} V_c(x) + \frac{3b}{7c^2} e^{\frac{7}{3} c^2 \gamma} + b_c \right). \quad (30)$$

Substituting (29) and (30) into (28) yields

$$I_1 \leq 2C^{1/2} \gamma^{3/2} (1 + \gamma^{-1})^{1/2} \left(e^{-\frac{7}{3} c^2 q_\gamma k_\gamma \gamma} V_c(x) + \frac{3b}{7c^2} e^{\frac{7}{3} c^2 \gamma} + b_c \right) \leq C(\lambda^{n\gamma} V_c(x) + \gamma),$$

where $\lambda \in (0, 1)$. By using similar arguments to I_2 , one obtains (5). Finally, by sending n to infinity, (6) can be obtained.

4 Lipschitz case

In the context of a Lipschitz gradient, assume **H3** - **H6** hold. Then, by **H4** and **H5**, one obtains, for any $x, y \in \mathbb{R}^d$

$$|\nabla^2 U(x)y| \leq L_1 |y|, \quad |\vec{\Delta}(\nabla U(x))| \leq \sqrt{d} L_2. \quad (31)$$

Moreover, in the Lipschitz case, there is no need to use the tamed coefficients. Thus, the linear interpolation of the scheme becomes

$$\bar{x}_t = \bar{x}_0 - \int_0^t \nabla \tilde{U}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) ds + \sqrt{2} w_t, \quad (32)$$

for all $t \geq 0$, where

$$\nabla \tilde{U}(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = \nabla U(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) + \nabla U_1(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) + \nabla U_2(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}),$$

with

$$\nabla U_1(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = - \int_{\lfloor s/\gamma \rfloor \gamma}^s \left(\nabla^2 U(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) \nabla U(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) - \vec{\Delta}(\nabla U)(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) \right) dr,$$

$$\nabla U_2(s, \bar{x}_{\lfloor s/\gamma \rfloor \gamma}) = \sqrt{2} \int_{\lfloor s/\gamma \rfloor \gamma}^s \nabla^2 U(\bar{x}_{\lfloor s/\gamma \rfloor \gamma}) dw_r.$$

One notes that for any $n \in \mathbb{N}$, $X_n = \bar{x}_{n\gamma}$.

4.1 Moment bounds

Proposition 3. Assume **H3** - **H6** are satisfied. Let x^* be the unique minimizer of U . Then, for all $x \in \mathbb{R}^d$, $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2+2L_2^2} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$ and $n \in \mathbb{N}$,

$$\mathbb{E}^{\mathcal{F}_0} |\bar{x}_{(n+1)\gamma} - x^*|^2 \leq q_1^{n+1} |\bar{x}_0 - x^*|^2 + \frac{q_2}{1 - q_1},$$

where $q_1 = 1 + \frac{19}{4}\gamma^2 L_1^2 + \frac{1}{2}\gamma^2 L_2^2 - 2m\gamma$ and $q_2 = \frac{5}{4}\gamma^2 d + 4d\gamma$

Proof. By using the scheme (32), one obtains

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - x^*|^2 \\ &= \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^* + \bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2 \\ &= \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^*|^2 + \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2 + 2\mathbb{E}^{\mathcal{F}_{n\gamma}} (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}). \end{aligned} \quad (33)$$

where

$$\begin{aligned} \bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma} &= -\nabla U(\bar{x}_{n\gamma})\gamma + \frac{\gamma^2}{2} \left(\nabla^2 U(\bar{x}_{n\gamma}) \nabla U(\bar{x}_{n\gamma}) - \tilde{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) \\ &\quad - \sqrt{2} \int_{n\gamma}^{(n+1)\gamma} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s dr + \sqrt{2} \int_{n\gamma}^{(n+1)\gamma} dw_r. \end{aligned}$$

Expanding the square of $|\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2$ gives

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2 \\ &= \left| -\nabla U(\bar{x}_{n\gamma})\gamma + \frac{\gamma^2}{2} \left(\nabla^2 U(\bar{x}_{n\gamma}) \nabla U(\bar{x}_{n\gamma}) - \tilde{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) \right|^2 \\ &\quad + 2\mathbb{E}^{\mathcal{F}_{n\gamma}} \left| \int_{n\gamma}^{(n+1)\gamma} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s dr \right|^2 + 2d\gamma - 2\gamma^2 \text{tr}(\nabla^2 U(\bar{x}_{n\gamma})) \\ &\leq 3\gamma^2 L_1^2 |\bar{x}_{n\gamma} - x^*|^2 + \frac{3}{4}\gamma^4 L_1^4 |\bar{x}_{n\gamma} - x^*|^2 + \frac{3}{4}\gamma^4 dL_2^2 + 2L_1^2 d\gamma^3 + 2d\gamma, \end{aligned} \quad (34)$$

where the last inequality holds due to (31) and the strong convexity of U . Then, by using **H3** and (31), one obtains

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}) \\ &\leq -m\gamma |\bar{x}_{n\gamma} - x^*|^2 + \frac{\gamma^2}{2} L_1^2 |\bar{x}_{n\gamma} - x^*|^2 + \frac{\gamma^2}{4} d + \frac{\gamma^2}{4} L_2^2 |\bar{x}_{n\gamma} - x^*|^2. \end{aligned} \quad (35)$$

Substituting (34) and (35) into (33) yields

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - x^*|^2 &\leq \left(1 + 4\gamma^2 L_1^2 + \frac{3}{4}\gamma^4 L_1^4 + \frac{\gamma^2}{2} L_2^2 - 2m\gamma \right) |\bar{x}_{n\gamma} - x^*|^2 \\ &\quad + \frac{3}{4}\gamma^4 dL_2^2 + 2L_1^2 d\gamma^3 + 2d\gamma + \frac{\gamma^2}{2} d. \end{aligned}$$

Finally, denote by $q_1 = 1 + \frac{19}{4}\gamma^2 L_1^2 + \frac{1}{2}\gamma^2 L_2^2 - 2m\gamma$ and $q_2 = \frac{5}{4}\gamma^2 d + 4d\gamma$, the result can be obtained by induction with $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2+2L_2^2} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. \square

Proposition 4. Assume **H3** - **H6** are satisfied. Let x^* be the unique minimizer of U . Then, for all $x \in \mathbb{R}^d$, $\gamma \in \left(0, 1 \wedge \frac{4m}{384L_1^4+2L_1^2+L_2^2+14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$ and $n \in \mathbb{N}$,

$$\mathbb{E}^{\mathcal{F}_0} |\bar{x}_{(n+1)\gamma} - x^*|^4 \leq q_3^{n+1} |\bar{x}_0 - x^*|^4 + \frac{q_4}{1 - q_3},$$

where $q_3 = 1 + 14\gamma^2 + 2\gamma^2 L_1^2 + \gamma^2 L_2^2 + 284\gamma^2 L_1^4 - 4m\gamma$, and $q_4 = 6144d^2 + 129\gamma^2 d^2$.

Proof. By using the scheme (3), one obtains the following expression

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - x^*|^4 \\ &= \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^* + \bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^4 \\ &= |\bar{x}_{n\gamma} - x^*|^4 + \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^4 + 6\mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^*|^2 |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2 \\ &\quad + 4\mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^*|^2 (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}) \\ &\quad + 4\mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^2 (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}). \end{aligned}$$

where

$$\begin{aligned} \bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma} &= -\nabla U(\bar{x}_{n\gamma})\gamma + \frac{\gamma^2}{2} \left(\nabla^2 U(\bar{x}_{n\gamma}) \nabla U(\bar{x}_{n\gamma}) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) \\ &\quad - \sqrt{2} \int_{n\gamma}^{(n+1)\gamma} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s dr + \sqrt{2} \int_{n\gamma}^{(n+1)\gamma} dw_r. \end{aligned}$$

The above expression can be further estimated as

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - x^*|^4 &\leq (1 + 13\gamma^2) |\bar{x}_{n\gamma} - x^*|^4 + \left(2 + \frac{2}{\gamma^2}\right) \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^4 \\ &\quad + 4\mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^*|^2 (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}). \end{aligned} \quad (36)$$

One observes that, by using **H4** and (31)

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}|^4 &\leq 64\gamma^4 L_1^4 |\bar{x}_{n\gamma} - x^*|^4 + 768L_1^4 \gamma^6 d^2 + 768\gamma^2 d^2 \\ &\quad + 4\gamma^8 |\nabla^2 U(\bar{x}_{n\gamma}) \nabla U(\bar{x}_{n\gamma}) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})|^4 \\ &\leq (64\gamma^4 L_1^4 + 32\gamma^8 L_1^8) |\bar{x}_{n\gamma} - x^*|^4 \\ &\quad + 768L_1^4 \gamma^6 d^2 + 768\gamma^2 d^2 + 32\gamma^8 d^2 L_2^4. \end{aligned} \quad (37)$$

It can be shown that due to **H3**

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{n\gamma} - x^*|^2 (\bar{x}_{n\gamma} - x^*)(\bar{x}_{(n+1)\gamma} - \bar{x}_{n\gamma}) \\ &\leq \left(\frac{\gamma^2}{2} L_1^2 + \frac{\gamma^2}{4} L_2^2 + \frac{\gamma^2}{4} - m\gamma \right) |\bar{x}_{n\gamma} - x^*|^4 + \frac{\gamma^2}{16} d^2. \end{aligned} \quad (38)$$

Then, substitute (37) and (38) into (36), one obtains

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_{(n+1)\gamma} - x^*|^4 &\leq (1 + 14\gamma^2 + 2\gamma^2 L_1^2 + \gamma^2 L_2^2 + 256\gamma^2 L_1^4 + 128\gamma^6 L_1^8 - 4m\gamma) |\bar{x}_{n\gamma} - x^*|^4 \\ &\quad + 3072L_1^4 \gamma^4 d^2 + 3072d^2 + 128\gamma^6 d^2 L_2^4 + \frac{\gamma^2}{4} d^2 L_2^4. \end{aligned}$$

Denote by $q_3 = 1 + 14\gamma^2 + 2\gamma^2 L_1^2 + \gamma^2 L_2^2 + 384\gamma^2 L_1^4 - 4m\gamma$, $q_4 = 6144d^2 + 129\gamma^2 d^2$, and the proof completes by induction with $\gamma \in \left(0, 1 \wedge \frac{4m}{384L_1^4+2L_1^2+L_2^2+14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. \square

4.2 Proof of Theorem 3

The explicit constants for the second and the fourth moments are obtained, then by using the following lemmas, one can show the rate of convergence in Wasserstein distance.

Lemma 8. *Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2+2L_2^2} \wedge \frac{4m}{384L_1^4+2L_1^2+L_2^2+14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. Then, for all $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,*

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_1(t, \bar{x}_{n\gamma})|^2] \leq 2\gamma^2(L_1^4|\bar{x}_{n\gamma} - x^*|^2 + dL_2^2), \quad \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_2(t, \bar{x}_{n\gamma})|^2] \leq 2\gamma dL_1^2,$$

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_1(t, \bar{x}_{n\gamma})|^4] \leq 8\gamma^4(L_1^8|\bar{x}_{n\gamma} - x^*|^4 + d^2L_2^4), \quad \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_2(t, \bar{x}_{n\gamma})|^4] \leq 12L_1^4d^2\gamma^2.$$

Proof. The proof is straightforward by using (31). \square

Lemma 9. *Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2+2L_2^2} \wedge \frac{4m}{384L_1^4+2L_1^2+L_2^2+14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. Then, for all $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,*

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_t - \bar{x}_{n\gamma}|^2] \leq \gamma(c_1|\bar{x}_{n\gamma} - x^*|^2 + c_2),$$

where $c_1 = \frac{15}{4}L_1$, $c_2 = 4d + \frac{3}{4}\gamma d$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_t - \bar{x}_{n\gamma}|^4] \leq \gamma^2(c_3|\bar{x}_{n\gamma} - x^*|^4 + c_4),$$

where $c_3 = 96L_1^2$, $c_4 = 768d^2 + 800\gamma^2d^2$, and

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|x_t - x_{n\gamma}|^2] \leq 2\gamma^2L_1^2|x_{n\gamma} - x^*|^2 + 8\gamma d.$$

Proof. One observes that the first two results can be obtained immediately by using (34) and (37). As for the third result, consider

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|x_t - x_{n\gamma}|^2] &= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| -\int_{n\gamma}^t \nabla U(x_r) dr + \sqrt{2} \int_{n\gamma}^t dw_r \right|^2 \right] \\ &\leq 2\gamma L_1^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} |x_r - x^*|^2 dr + 4\gamma d \\ &\leq 2\gamma^2 L_1^2 |x_{n\gamma} - x^*|^2 + 4\gamma^3 L_1^2 d + 4\gamma d \\ &\leq 2\gamma^2 L_1^2 |x_{n\gamma} - x^*|^2 + 8\gamma d, \end{aligned}$$

where the last inequality holds by using Theorem 1 in [11]. \square

Lemma 10. *Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2+2L_2^2} \wedge \frac{4m}{384L_1^4+2L_1^2+L_2^2+14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. Then, for all $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,*

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma})|^2] \\ \leq \gamma^2(c_5|\bar{x}_{n\gamma} - x^*|^4 + c_6|\bar{x}_{n\gamma} - x^*|^2 + c_7), \end{aligned}$$

where c_5, c_6 and c_7 are given explicitly in the proof.

Proof. For any $t \in [n\gamma, (n+1)\gamma)$, applying Itô's formula to $\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma})$ gives, almost surely

$$\begin{aligned} & \nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma}) \\ &= - \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U(\bar{x}_{n\gamma}) dr - \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_1(r, \bar{x}_{n\gamma}) + \nabla U_2(r, \bar{x}_{n\gamma})) dr \\ & \quad + \sqrt{2} \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r + \int_{n\gamma}^t \left(\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right) dr = \sum_{i=1}^4 J_i(t) \end{aligned} \quad (39)$$

Then, squaring both sides and taking conditional expectation gives

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma})|^2 \right] \leq 4 \sum_{i=1}^4 \mathbb{E}^{\mathcal{F}_{n\gamma}} [J_i(t)^2]. \quad (40)$$

By using Cauchy-Schwarz inequality, **H4**, **H5** and Lemma 9, one obtains

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [J_1(t)^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|(\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U(\bar{x}_{n\gamma})|^2] dr \\ &\leq \gamma L_1^2 L_2^2 |\bar{x}_{n\gamma} - x^*|^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_r - \bar{x}_{n\gamma}|^2 dr \\ &\leq \gamma^3 (c_1 L_1^2 L_2^2 |\bar{x}_{n\gamma} - x^*|^4 + c_2 L_1^2 L_2^2 |\bar{x}_{n\gamma} - x^*|^2). \end{aligned}$$

Similarly, by Cauchy-Schwarz inequality, (31) and Lemma 8, we have

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [J_2(t)^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla^2 U(\bar{x}_r) (\nabla U_1(r, \bar{x}_{n\gamma}) + \nabla U_2(r, \bar{x}_{n\gamma}))|^2] dr \\ &\leq 2\gamma L_1^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U_1(r, \bar{x}_{n\gamma})|^2 + |\nabla U_2(r, \bar{x}_{n\gamma})|^2] dr \\ &\leq 2\gamma^2 L_1^2 (2\gamma^2 (L_1^4 |\bar{x}_{n\gamma} - x^*|^2 + dL_2^2) + 2\gamma dL_1^2) \\ &\leq \gamma^3 (4\gamma L_1^6 |\bar{x}_{n\gamma} - x^*|^2 + 4\gamma L_1^2 L_2^2 d + 4dL_1^4). \end{aligned}$$

Moreover, applying Cauchy-Schwarz inequality, **H5** and Lemma 9 yields

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [J_3(t)^2] &\leq 2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})|^2] dr \\ &\leq 2L_2^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_r - \bar{x}_{n\gamma}|^2 dr \\ &\leq \gamma^2 (2L_2^2 c_1 |\bar{x}_{n\gamma} - x^*|^2 + 2L_2^2 c_2). \end{aligned}$$

Furthermore, one obtains by using Cauchy-Schwarz inequality, **H6** and Lemma 9

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} [J_4(t)^2] &\leq \gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma}) \right|^2 \right] dr \\ &\leq dL\gamma \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} |\bar{x}_r - \bar{x}_{n\gamma}|^2 dr \\ &\leq \gamma^3 (dLc_1 |\bar{x}_{n\gamma} - x^*|^2 + dLc_2). \end{aligned}$$

The proof completes by substituting all the estimates above into (40), i.e.

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma})|^2] \\ & \leq \gamma^2(c_5|\bar{x}_{n\gamma} - x^*|^4 + c_6|\bar{x}_{n\gamma} - x^*|^2 + c_7), \end{aligned}$$

where $c_5 = 15L_1^2L_2^2$, $c_6 = d(16L_1L_2^2 + 3L_2^2 + 15L) + 30L_1L_2^2 + 16L_1^4$ and $c_7 = d^2(16 + 3\gamma) + d(16L_1^3 + 48L_2^2 + 6L_2)$. \square

In the following proofs, denote by $M(x, \bar{x})$ a matrix whose (i, j) -th entry is $\sum_{k=1}^d \frac{\partial^3 U(\bar{x})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (x^{(k)} - \bar{x}^{(k)})$, for any $x, \bar{x} \in \mathbb{R}^d$.

Lemma 11. Assume **H6** holds. Then, for any $x, \bar{x} \in \mathbb{R}^d$, and $i = 1, \dots, d$,

$$|\nabla^2 U(x) - \nabla^2 U(\bar{x}) - M(x, \bar{x})| \leq L|x - \bar{x}|^2.$$

Proof. The proof follows the same lines as Lemma 4, hence omitted. \square

Lemma 12. Assume **H3** and **H5** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2 + 2L_2^2} \wedge \frac{4m}{384L_1^4 + 2L_1^2 + L_2^2 + 14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. Then, for all $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right|^2 \right] \leq \gamma^2(L_2^2 c_1 |\bar{x}_{n\gamma} - x^*|^2 + L_2^2 c_2).$$

Proof. By using conditional Itô's isometry and Lemma 2, one obtains

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right|^2 \right] &= \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\sum_{i=1}^d \left| \sum_{j=1}^d \int_{n\gamma}^t M^{(i,j)}(\bar{x}_r, \bar{x}_{n\gamma}) dw_r^{(j)} \right|^2 \right] \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left| M^{(i,j)}(\bar{x}_r, \bar{x}_{n\gamma}) \right|^2 dr \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} \left| \sum_{k=1}^d \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (\bar{x}_r^{(k)} - \bar{x}_{n\gamma}^{(k)}) \right|^2 dr \\ &\leq L_2^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_r - \bar{x}_{n\gamma}|^2] dr \\ &\leq \gamma^2(L_2^2 c_1 |\bar{x}_{n\gamma} - x^*|^2 + L_2^2 c_2), \end{aligned}$$

where a detailed proof for the first inequality, which uses the mean value theorem is given in Appendix C. \square

Lemma 13. Assume **H3** - **H6** are satisfied. Let $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2 + 2L_2^2} \wedge \frac{4m}{384L_1^4 + 2L_1^2 + L_2^2 + 14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2}\right)$. Then, for all $n \in \mathbb{N}$, and $t \in [n\gamma, (n+1)\gamma)$,

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \leq \gamma^3(c_8|x_{n\gamma} - x^*|^2 + c_9|\bar{x}_{n\gamma} - x^*|^2 + c_{10}),$$

where the constants c_8, c_9 and c_{10} are given explicitly in the proof.

Proof. The proof follows the same lines as in Lemma 6, thus, the main focus here is to provide explicit constants. The second term in (19) can be estimated as

$$\begin{aligned}
& \sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \int_{n\gamma}^r (\nabla^2 U(x_{n\gamma}) - \nabla^2 U(\bar{x}_{n\gamma})) dw_s dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&= \sqrt{2} \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\sum_{i=1}^d \int_{n\gamma}^t \sum_{j=1}^d \int_{n\gamma}^r (\nabla^2 U^{(i,j)}(x_{n\gamma}) - \nabla^2 U^{(i,j)}(\bar{x}_{n\gamma})) dw_s^{(j)} dr \right. \\
&\quad \times \sum_{j=1}^d \int_{n\gamma}^t \sum_{k=1}^d \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \left(- \int_{n\gamma}^r \nabla \tilde{U}^{(k)}(s, \bar{x}_{n\gamma}) ds + \sqrt{2} \int_{n\gamma}^r dw_s^{(k)} \right) dw_r^{(j)} \left. \right] \\
&\leq \frac{1}{2} \sum_{i=1}^d \mathbb{E}^{\mathcal{F}_{n\gamma}} \left| \int_{n\gamma}^t \sum_{j=1}^d \int_{n\gamma}^r (\nabla^2 U^{(i,j)}(x_{n\gamma}) - \nabla^2 U^{(i,j)}(\bar{x}_{n\gamma})) dw_s^{(j)} dr \right|^2 \\
&\quad + \sum_{i=1}^d \mathbb{E}^{\mathcal{F}_{n\gamma}} \left| \sum_{j=1}^d \int_{n\gamma}^t \sum_{k=1}^d \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \left(- \int_{n\gamma}^r \nabla \tilde{U}^{(k)}(s, \bar{x}_{n\gamma}) ds \right) dw_r^{(j)} \right|^2 \\
&\leq \gamma^3 (L_2^2 |x_{n\gamma} - x^*|^2 + (L_2^2 + 3L_1^2 L_2^2 + 6\gamma^2 L_1^4 L_2^2) |\bar{x}_{n\gamma} - x^*|^2 + 6\gamma^2 L_2^4 d + 6\gamma L_1^2 L_2^2 d).
\end{aligned}$$

where the first inequality holds due to Young's inequality and the fact that for any $i, j, k = 1, \dots, d$

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t \int_{n\gamma}^r (\nabla^2 U^{(i,j)}(x_{n\gamma}) - \nabla^2 U^{(i,j)}(\bar{x}_{n\gamma})) dw_s^{(j)} dr \int_{n\gamma}^t \frac{\partial^3 U(\bar{x}_{n\gamma})}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} \int_{n\gamma}^r \sqrt{2} dw_s^{(k)} dw_r^{(j)} \right] = 0,$$

while the last inequality holds due to Young's inequality, the mean value theorem, Cauchy-Schwarz inequality and Lemma 8. By using Cauchy-Schwarz inequality, (19) becomes

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
&\leq \sqrt{\left| \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right|^2} \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma \int_{n\gamma}^t |\nabla U(x_r) - \nabla U(x_{n\gamma}) - (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})) \right. \right. \\
&\quad \left. \left. - \sqrt{2} \int_{n\gamma}^r \nabla^2 U(x_{n\gamma}) dw_s + \sqrt{2} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s|^2 dr \right] \right)^{1/2} \\
&\quad + \gamma^3 (L_2^2 |x_{n\gamma} - x^*|^2 + (L_2^2 + 3L_1^2 L_2^2 + 6\gamma^2 L_1^4 L_2^2) |\bar{x}_{n\gamma} - x^*|^2 + 6\gamma^2 L_2^4 d + 6\gamma L_1^2 L_2^2 d).
\end{aligned}$$

Then, to estimate the first term of (19), one applies Itô's formula to $\nabla U(x_r) - \nabla U(x_{n\gamma})$ and $\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})$ to obtain, almost surely

$$\begin{aligned}
& \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma \int_{n\gamma}^t |\nabla U(x_r) - \nabla U(x_{n\gamma}) - (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma})) \right. \right. \\
&\quad \left. \left. - \sqrt{2} \int_{n\gamma}^r \nabla^2 U(x_{n\gamma}) dw_s + \sqrt{2} \int_{n\gamma}^r \nabla^2 U(\bar{x}_{n\gamma}) dw_s|^2 dr \right] \right)^{1/2} \\
&\leq 2 \left(\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \left| \nabla^2 U(x_s) \nabla U(x_s) - \vec{\Delta}(\nabla U)(x_s) \right|^2 ds dr \right] \right. \\
&\quad \left. + \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \left| \nabla^2 U(\bar{x}_s) \nabla \tilde{U}(s, \bar{x}_{n\gamma}) - \vec{\Delta}(\nabla U)(\bar{x}_s) \right|^2 ds dr \right] \right)
\end{aligned}$$

$$\begin{aligned}
& + 2\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla^2 U(x_s) - \nabla^2 U(x_{n\gamma})|^2 \right] ds dr \\
& + 2\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|\nabla^2 U(\bar{x}_s) - \nabla^2 U(\bar{x}_{n\gamma})|^2 \right] ds dr \Big)^{1/2} \\
& \leq 2 \left(\gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} [2L_1^4 |x_s - x^*|^2 + 2L_2^2 d] ds dr \right. \\
& \quad + \gamma^2 \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[2L_1^2 |\nabla \tilde{U}(s, \bar{x}_{n\gamma})|^2 + 2L_2^2 d \right] ds dr \\
& \quad + 2\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} [L_2^2 |x_s - x_{n\gamma}|^2] ds dr \\
& \quad \left. + 2\gamma \int_{n\gamma}^t \int_{n\gamma}^r \mathbb{E}^{\mathcal{F}_{n\gamma}} [L_2^2 |\bar{x}_s - \bar{x}_{n\gamma}|^2] ds dr \right)^{1/2} \\
& \leq 2\gamma^2 (2L_1^4 |x_{n\gamma} - x^*|^2 + 4\gamma L_1^4 d + 2L_2^2 d \\
& \quad + (6L_1^4 + 12\gamma^2 L_1^6) |\bar{x}_{n\gamma} - x^*|^2 + 12\gamma^2 L_1^2 L_2^2 d + 12\gamma L_1^4 d + 2L_2^2 d \\
& \quad + 2(L_2^2 c_1 |\bar{x}_{n\gamma} - x^*|^2 + L_2^2 c_2) \\
& \quad + 2(2\gamma L_1^2 L_2^2 |x_{n\gamma} - x^*|^2 + 8L_2^2 d))^{1/2},
\end{aligned}$$

where the first inequality holds due to Cauchy-Schwarz inequality and Young's inequality, the second inequality holds by using (31) and **H5**, while the last inequality is obtained due to Lemma 8 and 9. Finally by using Young's inequality and Lemma 12, one obtains

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \leq \gamma^3 (c_8 |x_{n\gamma} - x^*|^2 + c_9 |\bar{x}_{n\gamma} - x^*|^2 + c_{10}),$$

where $c_8 = 2L_1^4 + 4L_1 L_2^2 + L_2^2$, $c_9 = 9L_1^2 L_2^2 + \frac{45}{4} L_1 L_2^2 + L_2^2 + 18L_1^4$ and $c_{10} = d(16L_1^3 + 6L_1 L_2^2 + 50L_2^2 + \frac{9}{4} L_2 d)$. \square

Proof of Theorem 3. Note that in the Lipschitz case, there are restrictions for the stepsize γ , i.e. $\gamma \in \left(0, 1 \wedge \frac{8m}{19L_1^2 + 2L_2^2} \wedge \frac{4m}{384L_1^4 + 2L_1^2 L_2^2 + 14} \wedge \frac{1}{L_1} \wedge \frac{1}{L_2} \wedge \frac{1}{L}\right)$. Consider the synchronous coupling of x_t and \bar{x}_t for $t \geq 0$, where \bar{x}_t is defined by (32). Let (x_0, \bar{x}_0) distributed according to ζ_0 , where $\zeta_0 = \pi \otimes \delta_x$ for all $x \in \mathbb{R}^d$. Define $e_t = x_t - \bar{x}_t$, for all $t \in [n\gamma, (n+1)\gamma)$, $n \in \mathbb{N}$. By Itô's formula, one obtains, almost surely,

$$|e_t|^2 = |e_{n\gamma}|^2 - 2 \int_{n\gamma}^t e_s (\nabla U(x_s) - \nabla \tilde{U}(s, \bar{x}_{n\gamma})) ds.$$

Then, taking the expectation conditional on $\mathcal{F}_{n\gamma}$ and taking the derivative on both sides yield

$$\begin{aligned}
\frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] &= -2 \mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_t (\nabla U(x_t) - \nabla \tilde{U}(t, \bar{x}_{n\gamma})) \right] \\
&= 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-(\nabla U(x_t) - \nabla U(\bar{x}_t)))] \\
&\quad + 2 \mathbb{E}^{\mathcal{F}_{n\gamma}} [e_t (-(\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma})))] .
\end{aligned}$$

By applying Itô's formula to $\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma})$, and by calculating $\nabla U(\bar{x}_t) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(t, \bar{x}_{n\gamma}) - \nabla U_2(t, \bar{x}_{n\gamma})$, one obtains (39). Substituting (39) into the above equation and by using **H3** yield

$$\frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq -2m \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2]$$

$$\begin{aligned}
& + 2\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma}) dr \right| \right] \\
& + 2\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr \right| \right] \\
& + 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_t \left(- \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) dw_r \right) \right] \\
& + 2\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[|e_t| \left| \int_{n\gamma}^t (\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})) dr \right| \right] \\
& \leq -(2m - 3\varepsilon)\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\
& + \frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma})) \nabla U_\gamma(\bar{x}_{n\gamma}) dr \right|^2 \right] \\
& + \frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t \nabla^2 U(\bar{x}_r) (\nabla U_{1,\gamma}(r, \bar{x}_{n\gamma}) + \nabla U_{2,\gamma}(r, \bar{x}_{n\gamma})) dr \right|^2 \right] \\
& + \frac{1}{\varepsilon}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\left| \int_{n\gamma}^t (\vec{\Delta}(\nabla U)(\bar{x}_r) - \vec{\Delta}(\nabla U)(\bar{x}_{n\gamma})) dr \right|^2 \right] \\
& - 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_t \int_{n\gamma}^t (\nabla^2 U(\bar{x}_r) - \nabla^2 U(\bar{x}_{n\gamma}) - M(\bar{x}_r, \bar{x}_{n\gamma})) dw_r \right] \\
& - 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[(e_t - e_{n\gamma}) \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] - 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[e_{n\gamma} \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right],
\end{aligned}$$

where the last inequality holds due to Young's inequality and the last term is zero. Then, by using the results in Lemma 9, 11 and by taking $\varepsilon = \frac{m}{4}$, one obtains

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] & \leq -m\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\
& + \frac{4}{m}\gamma^3(c_1L_1^2L_2^2|\bar{x}_{n\gamma} - x^*|^4 + c_2L_1^2L_2^2|\bar{x}_{n\gamma} - x^*|^2) \\
& + \frac{4}{m}\gamma^3(4\gamma L_1^6|\bar{x}_{n\gamma} - x^*|^2 + 4\gamma L_1^2L_2^2d + 4dL_1^4) \\
& + \frac{4}{m}\gamma^3(dLc_1|\bar{x}_{n\gamma} - x^*|^2 + dLc_2) \\
& + \frac{8}{m}L^2 \int_{n\gamma}^t \mathbb{E}^{\mathcal{F}_{n\gamma}} [|\bar{x}_r - \bar{x}_{n\gamma}|^4] dr \\
& + 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(x_r) - \nabla U(\bar{x}_r)) dr \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
& + 2\sqrt{2}\mathbb{E}^{\mathcal{F}_{n\gamma}} \left[\int_{n\gamma}^t (\nabla U(\bar{x}_r) - \nabla U(\bar{x}_{n\gamma}) - \nabla U_1(r, \bar{x}_{n\gamma}) - \nabla U_2(r, \bar{x}_{n\gamma})) dr \right. \\
& \quad \left. \times \int_{n\gamma}^t M(\bar{x}_r, \bar{x}_{n\gamma}) dw_r \right] \\
& \leq -m\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \\
& + \frac{4}{m}\gamma^3(c_1L_1^2L_2^2|\bar{x}_{n\gamma} - x^*|^4 + c_2L_1^2L_2^2|\bar{x}_{n\gamma} - x^*|^2) \\
& + \frac{4}{m}\gamma^3(4\gamma L_1^6|\bar{x}_{n\gamma} - x^*|^2 + 4\gamma L_1^2L_2^2d + 4dL_1^4) \\
& + \frac{4}{m}\gamma^3(dLc_1|\bar{x}_{n\gamma} - x^*|^2 + dLc_2)
\end{aligned}$$

$$\begin{aligned}
& + \frac{8}{m} L^2 \gamma^3 (c_3 |\bar{x}_{n\gamma} - x^*|^4 + c_4) \\
& + 2\sqrt{2} \gamma^3 (c_8 |x_{n\gamma} - x^*|^2 + c_9 |\bar{x}_{n\gamma} - x^*|^2 + c_{10}) \\
& + \gamma^3 (c_5 |\bar{x}_{n\gamma} - x^*|^4 + c_6 |\bar{x}_{n\gamma} - x^*|^2 + c_7) \\
& + 2\gamma^3 (L_2^2 c_1 |\bar{x}_{n\gamma} - x^*|^2 + L_2^2 c_2)
\end{aligned}$$

where the last inequality holds by using Cauchy-Schwarz inequality, Young's inequality and Lemma 9, 13, 10, 12. Then, after simplification, one obtains

$$\frac{d}{dt} \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq -m \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] + \gamma^3 (c_{11} |\bar{x}_{n\gamma} - x^*|^4 + c_{12} |\bar{x}_{n\gamma} - x^*|^2 + c_{13} |x_{n\gamma} - x^*|^2 + c_{14}),$$

where $c_{11} = \frac{15}{m} L_1^3 L_2^2 + 15 L_1^2 L_2^2 + \frac{768}{m} L_1^2 L^2$, $c_{12} = d(\frac{16}{m} L_1^2 L_2^2 + \frac{16m+3}{m} L_1 L_2^2 + \frac{15}{m} L_1 L + 3 L_2^2 + 15 L) + \frac{16}{m} L_1^5 + \frac{165}{2} L_1 L_2^2 + 88 L_1^4 + 36 L_1^2 L_2^2 + 4 L_2^2$, $c_{13} = 2\sqrt{2} c_8$ and $c_{14} = d^2(\frac{6144}{m} L^2 + \frac{16}{m} L + \frac{6403+19m}{m}) + d(\frac{16}{m} L_1^4 + 80 L_1^3 + \frac{24m+16}{m} L_1 L_2^2 + 256 L_2^2 + \frac{33}{2} L_2)$. Then, the application of Gronwall's lemma yields

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq e^{-m(t-n\gamma)} |e_{n\gamma}|^2 + \gamma^4 (c_{11} |\bar{x}_{n\gamma} - x^*|^4 + c_{12} |\bar{x}_{n\gamma} - x^*|^2 + c_{13} |x_{n\gamma} - x^*|^2 + c_{14}).$$

Finally, by induction, Proposition 3 and 4, one obtains

$$\begin{aligned}
& \mathbb{E} [|e_{(n+1)\gamma}|^2] \\
& = \mathbb{E} \left[\mathbb{E}^{\mathcal{F}_0} \left[\dots \mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_{(n+1)\gamma}|^2] \dots \right] \right] \\
& \leq e^{-m\gamma(n+1)} \mathbb{E} [|e_0|^2] + \frac{\gamma^4 c_{14}}{1 - e^{-m\gamma}} + \gamma^4 c_{11} \sum_{k=0}^n \mathbb{E} [|\bar{x}_{k\gamma} - x^*|^4] e^{-m\gamma(n-k)} \\
& \quad + \gamma^4 c_{12} \sum_{k=0}^n \mathbb{E} [|\bar{x}_{k\gamma} - x^*|^2] e^{-m\gamma(n-k)} + \gamma^4 c_{13} \sum_{k=0}^n \mathbb{E} [|x_{k\gamma} - x^*|^2] e^{-m\gamma(n-k)} \\
& \leq e^{-m\gamma(n+1)} \mathbb{E} [|x_0 - \bar{x}_0|^2] + \frac{\gamma^3 e^{m\gamma}}{m} \left(c_{14} + \frac{c_{11} q_4}{1 - q_3} + \frac{c_{12} q_2}{1 - q_1} + 2c_{13} d\gamma \right) \\
& \quad + \gamma^3 \left(\frac{c_{11} \mathbb{E} [|\bar{x}_0 - x^*|^4] Q_3^{-1} \bar{Q}_3^{n+1}}{|m + \gamma^{-1} \ln q_3|} + \frac{c_{12} \mathbb{E} [|\bar{x}_0 - x^*|^2] Q_1^{-1} \bar{Q}_1^{n+1}}{|m + \gamma^{-1} \ln q_1|} + \frac{c_{13} \mathbb{E} [|x_0 - x^*|^2] e^{-m\gamma(n-1)}}{m} \right),
\end{aligned}$$

where $Q_3 = \min(e^{-m\gamma}, q_3)$, $\bar{Q}_3 = \max(e^{-m\gamma}, q_3)$, $Q_1 = \min(e^{-m\gamma}, q_1)$, $\bar{Q}_1 = \max(e^{-m\gamma}, q_1)$ and the last inequality holds by using $1 - e^{-m\gamma} \geq m\gamma e^{-m\gamma}$. The application of Theorem 1 in [11] with the initial distribution ζ_0 completes the proof.

Proof of Corollary 1. In the case that the target distribution π is a multivariate Gaussian distribution, by using the same arguments, one obtains the following bound

$$\mathbb{E}^{\mathcal{F}_{n\gamma}} [|e_t|^2] \leq e^{-m(t-n\gamma)} |e_{n\gamma}|^2 + \gamma^4 \left(\frac{4}{m} L_1^5 |\bar{x}_{n\gamma} - x^*|^2 + 4L_1 L_2^2 d + 4dL_1^4 \right),$$

which indicates

$$W_2^2(\delta_x R_\gamma^n, \pi) \leq e^{-mn\gamma} \left(2|x - x^*|^2 + \frac{2d}{m} \right) + \tilde{C} \gamma^3,$$

where $\tilde{C} = \frac{e^{m\gamma}}{m} \left(4L_1 L_2^2 d + 4dL_1^4 + \frac{4L_1^5 q_2}{m(1-q_1)} \right) + \left(\frac{4L_1^5 |x - x^*|^2 Q_1^{-1} \bar{Q}_1^{n+1}}{m|m + \gamma^{-1} \ln q_1|} \right)$.

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Appendices

Appendix A

In order to prove (8), one needs the following definition and the propositions.

Definition 1. Consider a probability measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu)$. Let \mathcal{A} be the set of continuously differentiable, Lipschitz functions on \mathbb{R}^d . We say that ν satisfies a Log-Sobolev inequality if there exists $C > 0$ such that

$$\text{Ent}_\nu(f^2) \leq 2C \int_{\mathbb{R}^d} |\nabla f|^2 d\nu,$$

for every function $f \in \mathcal{A}$ with $\text{Ent}_\nu(f^2 \log^+ f^2) < \infty$, where

$$\text{Ent}_\nu(f) = \mathbb{E}_\nu(f \log f) - \mathbb{E}_\nu(f) \log \mathbb{E}_\nu(f).$$

For more details about the definition of the Log-Sobolev inequality, please refer to Chapter 2 in [6].

Proposition 5 (Proposition 5.4.1 in [5]). *If ν satisfies a logarithmic Sobolev inequality with constant $C > 0$, then for every 1-Lipschitz function f and every $\alpha^2 < 1/C$,*

$$\int_{\mathbb{R}^d} e^{\alpha^2 f^2/2} d\nu < \infty.$$

More precisely, any 1-Lipschitz function f is integrable and for every $s \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} e^{sf} d\nu < e^{s \int_{\mathbb{R}^d} f d\nu + Cs^2/2}.$$

Proposition 6 (Proposition 5.5.1 in [5]). *The standard Gaussian measure ν on the Borel sets of \mathbb{R}^d satisfies, for every $f \in \mathcal{A}$,*

$$\text{Ent}_\nu(f^2) \leq 2 \int_{\mathbb{R}^d} |\nabla f|^2 d\nu.$$

Proposition 6 implies that, for a Gaussian measure ν with mean μ and covariance matrix Q , by using change of variables, one obtains for every $f \in \mathcal{A}$ on \mathbb{R}^d ,

$$\text{Ent}_\nu(f^2) \leq 2 \int_{\mathbb{R}^d} (Q \nabla f) \nabla f d\nu. \quad (41)$$

One notes that the scheme (2) shows that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, conditional on the previous step $\bar{X}_{n-1} = x$, \bar{X}_n is a Gaussian random variable with mean $\mu(x) = x + \mu_\gamma(x)\gamma$ and covariance matrix $Q(x) = \text{diag} \left(\left(|\sigma_\gamma^{(k)}(x)|^2 \gamma \right)_{k \in 1, \dots, d} \right)$, where

$$\begin{aligned} \mu_\gamma(x) &= -\nabla U_\gamma(x) + \frac{\gamma}{2} \left((\nabla^2 U \nabla U)_\gamma(x) - \vec{\Delta}(\nabla U)_\gamma(x) \right), \\ \sigma_\gamma^{(k)}(x) &= \sqrt{2 + 2/3\gamma^2 \sum_{j=1}^d |\nabla^2 U_\gamma^{(k,j)}(x)|^2 - 2\gamma \nabla^2 U_\gamma^{(k,k)}(x)}. \end{aligned}$$

Then, by using (41), one obtains

$$\text{Ent}_\nu(f^2) \leq 2 \int_{\mathbb{R}^d} (Q \nabla f) \nabla f d\nu = 2 \int_{\mathbb{R}^d} \sum_{k=1}^d |\sigma_\gamma^{(k)}(x)|^2 |\nabla f^{(k)}(x)|^2 d\nu \leq 2 \int_{\mathbb{R}^d} \frac{14}{3} |\nabla f|^2 d\nu.$$

Therefore, applying Proposition 5 with $s = a$, $f = \sqrt{1 + |x|^2}$ and $C = \frac{14}{3}$ yields the desired result, i.e.

$$R_\gamma V_a(x) = \mathbb{E}_x(V_a(\bar{X}_1)) \leq e^{\frac{7}{3}\gamma a^2} \exp \left\{ a \mathbb{E}((1 + |\bar{X}_1|^2)^{1/2} | \bar{X}_0 = x) \right\}.$$

Appendix B

To obtain (25), one consider the following cases

(i) If $m > \frac{7}{3}c^2$,

$$\begin{aligned} C\gamma^{3+\beta} \mathbb{E}[V_c(\bar{x}_0)] \sum_{k=0}^n e^{-\frac{7}{3}c^2\gamma k - m\gamma(n-k)} &= C\gamma^{3+\beta} e^{-m\gamma n} \mathbb{E}[V_c(\bar{x}_0)] \sum_{k=0}^n e^{-\frac{7}{3}c^2\gamma k + m\gamma k} \\ &= C\gamma^{3+\beta} e^{-m\gamma n} \mathbb{E}[V_c(\bar{x}_0)] \frac{e^{(n+1)(m-\frac{7}{3}c^2)\gamma} - 1}{e^{(m-\frac{7}{3}c^2)\gamma} - 1} \\ &\leq C\gamma^{3+\beta} e^{-m\gamma n} \mathbb{E}[V_c(\bar{x}_0)] \frac{e^{n(m-\frac{7}{3}c^2)\gamma}}{1 - e^{-(m-\frac{7}{3}c^2)\gamma}} \\ &\leq \frac{C\mathbb{E}[V_c(\bar{x}_0)]}{m - \frac{7}{3}c^2} e^{m\gamma} \gamma^{2+\beta} e^{-\frac{7}{3}c^2(n+1)\gamma}. \end{aligned}$$

(ii) For the case $m < \frac{7}{3}c^2$, we have

$$\begin{aligned} C\gamma^{3+\beta}e^{-m\gamma n}\mathbb{E}[V_c(\bar{x}_0)]\sum_{k=0}^ne^{-\frac{7}{3}c^2\gamma k+m\gamma k} &\leq C\gamma^{3+\beta}e^{-m\gamma n}\mathbb{E}[V_c(\bar{x}_0)]\frac{1}{1-e^{-(\frac{7}{3}c^2-m)\gamma}} \\ &\leq \frac{C\mathbb{E}[V_c(\bar{x}_0)]}{\frac{7}{3}c^2-m}e^{\frac{7}{3}c^2\gamma}\gamma^{2+\beta}e^{-m(n+1)\gamma}. \end{aligned}$$

(iii) As for the case $m = \frac{7}{3}c^2$, it can be shown that

$$C\gamma^{3+\beta}e^{-m\gamma n}\mathbb{E}[V_c(\bar{x}_0)]\sum_{k=0}^ne^{-\frac{7}{3}c^2\gamma k+m\gamma k} = C(n+1)\gamma^{3+\beta}e^{-m\gamma n}\mathbb{E}[V_c(\bar{x}_0)] \leq \frac{C\mathbb{E}[V_c(\bar{x}_0)]}{m}e^{m\gamma}\gamma^{2+\beta}.$$

Appendix C

For all $x, y \in \mathbb{R}^d$ and a constant $c > 0$, denote by $g(t) = \nabla^2 U(x + tc(y - x))$. For any $i, j = 1, \dots, d$, $(g^{(i,j)})'(t) = c \sum_{k=1}^d \frac{\partial^3 U(x + t(y-x))}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (y^{(k)} - x^{(k)})$. Then, since

$$\nabla^2 U^{(i,j)}(x + c(y - x)) - \nabla^2 U^{(i,j)}(x) = g^{(i,j)}(1) - g^{(i,j)}(0),$$

by mean value theorem, there exists $t_c \in [0, 1]$, such that

$$\begin{aligned} |\nabla^2 U(x + c(y - x)) - \nabla^2 U(x)| &= |g(1) - g(0)| \\ &= c \sqrt{\sum_{i,j=1}^d \left| \sum_{k=1}^d \frac{\partial^3 U(x + t_c c(y - x))}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (y^{(k)} - x^{(k)}) \right|^2} \\ &\leq L_2 |c(y - x)|, \end{aligned}$$

which, by sending c to zero yields

$$\sqrt{\sum_{i,j=1}^d \left| \sum_{k=1}^d \frac{\partial^3 U(x)}{\partial x^{(i)} \partial x^{(j)} \partial x^{(k)}} (y^{(k)} - x^{(k)}) \right|^2} \leq L_2 |y - x|.$$